RECURRENCE RELATION

Def: A recurrence relation for the sequence is an equation that expresses a_n in terms of one or more of the previous terms of the sequence namely $a_0, a_1, \ldots, a_{n-1}$ for all integers with non-negative.

Example: The Fibonacci sequence is defined by the relation $F_n = F_{n-1} + F_{n-2}$; $F_1 = 1$, $F_2 = 1$ where F_1, F_2 are initial Condn

SOLUTION OF RECURRENCE RELATION

- 2M — ITERATION METHOD
- 8M — CHARACTERISTIC METHOD
- 2M, 8M — GENERATING METHOD

PROBLEMS UNDER ITERATION METHOD

① Find the recurrence relation and basis for the Seq $(1, 3, 3^2, \ldots)$

Sol: Let $a_0 = 1 = 3^0$, $a_1 = 3$, $a_2 = 3^2$, ..., $a_n = 3^n = 3 \cdot 3^{n-1} = 3a_{n-1}$.

$\therefore \boxed{a_n = 3a_{n-1}}$ is the recurrence relation

② Find the recurrence relation for the sequence $S(K) = 2K + 9$.

Sol: $S(K) = 2K+9$, $S(K-1) = 2(K-1) + 9 = 2K + 7$, $S(K) = 2K + 7 + 2$

$\therefore S_K = S(K-1) + 2 \Rightarrow \boxed{S(K) - S(K-1) = 2}$ is the recurrence relation.

③ Find the recurrence relation for the sequence $D(K) = 5 \cdot 2^K$

Sol: $D(K) = 5 \cdot 2^K$; $D(K-1) = 5 \cdot 2^{K-1}$.

Now $D(K) = 5 \cdot 2^{K-1} \cdot 2 = D(K-1) \cdot 2$.

$\therefore \boxed{D(K) - 2D(K-1) = 0}$ is the recurrence relation.

④ Find the recurrence relation for $y_n = A(3^n) + B(-4)^n$.

Sol: $\boxed{y_n = A3^n + B(-4)^n} \rightarrow ①$

$y_{n+1} = A3^{n+1} + B(-4)^{n+1}$

$\boxed{y_{n+1} = 3A3^n + (-4)B(-4)^n} \rightarrow ②$

$y_{n+2} = A3^{n+2} + B(-4)^{n+2}$

$\boxed{y_{n+2} = 3^2 A3^n + (-4)^2 B(-4)^n} \rightarrow 3$

③ + ② − 12①.

$\Rightarrow y_{n+2} + y_{n+1} - 12y_n = 9\cancel{A3^n} + 16\cancel{B(-4)^n} + 3\cancel{A3^n} - 4\cancel{B(-4)^n} - 12\cancel{A3^n} - 12\cancel{B(-4)^n} = 0$

$\therefore \boxed{y_{n+2} + y_{n+1} - 12y_n = 0}$ is the reqd recurrence relation

⑤ Find the recurrence relation for the relation $S(K) = K^2 - K$.

Sol $\boxed{S(K) = K^2 - K} \rightarrow ①$

$S(K-1) = (K-1)^2 - (K-1) = K^2 - 3K + 2$

$\rightarrow \boxed{S(K-1) = K^2 - 3K + 2} \rightarrow$ (2)

$S(K-2) = (K-2)^2 - (K-2) = K^2 - 4K + 4 - K + 2 = K^2 - 5K + 6$

$\rightarrow \boxed{S(K-2) = K^2 - 5K + 6} \rightarrow$ (3)

Now (1) $- 2 \times$ (2) $+$ (3)

$S(K) - 2S(K-1) + S(K-2) = K^2 - K - 2K^2 + 6K - 4 + K^2 - 5K + 6 = 2$

$\therefore \boxed{S(K) - 2S(K-1) + S(K-2) = 2}$ is reqd recurrence relation

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6) Solve the recurrence relation $a_n = a_{n-1} + (n-1), n \geq 2$ & $a_1 = 0$.

Sol: Given: $a_n = a_{n-1} + (n-1)$

$= a_{n-2} + (n-2) + (n-1)$

$= a_{n-3} + (n-3) + (n-2) + (n-1)$

$\vdots$

$= a_1 + 1 + 2 + 3 + \cdots + n$

$= 0 + \frac{(n-1)(n-1+1)}{2}$ $\quad [\because 1 + 2 + \cdots + n = \frac{n(n+1)}{2}]$

$\boxed{a_n = \frac{n(n-1)}{2}}, n \geq 1$ & $a_1 = 0$

We prove this by mathematical induction.

Let $P(n)$: $a_n = \frac{n(n-1)}{2} \rightarrow$ (1)

T.P $P(n)$ IS TRUE FOR $n \geq 1$

STEP 1 BASIS STEP : TO VERIFY $P(1)$ IS TRUE

Put $n=1$, in (1) $\quad P(1) = a_1 = \frac{1(1-1)}{2} \Rightarrow 0 = 0$ IS TRUE $\quad [\because a_1 = 0]$

$\therefore P(1)$ IS TRUE.

STEP 2 : INDUCTIVE STEP : Assume $P(K)$ IS TRUE, $K \geq 1$.

Put $n = K$ in (1), $\quad P(K)$: $a_K = \frac{K(K-1)}{2}$ IS TRUE $\rightarrow$ (2)

STEP 3 : T.P $P(K+1)$ IS TRUE

ie) T.P $\quad P(K+1)$: $a_{K+1} = \frac{(K+1)K}{2}$, $K \geq 1$.

$P(K+1)$: $a_{K+1} = a_K + K$ $\quad$ [By recurrence formula].

$= \frac{K(K-1)}{2} + K$ $\quad$ [From 2].

$= \frac{K^2 - K + 2K}{2}$

$= \frac{K^2 + K}{2}$

$a_{K+1} = \frac{K(K+1)}{2}$

$\therefore P(K+1)$ IS TRUE.

Thus $P(K)$ IS TRUE $\Rightarrow P(K+1)$ IS TRUE.

$\therefore P(n)$ is true $\forall n \geq 1 \Rightarrow a_n = \frac{n(n-1)}{2}, \forall n \geq 1$.

(7) Solve the recurrence relation $a_n - a_{n-1} = 2$, $n \geq 2$ & $a_1 = 2$ by Iteration method.

Sol: Given $a_n - a_{n-1} = 2$

$a_{n-1} - a_{n-2} = 2$

$a_{n-3} - a_{n-4} = 2$

$\vdots$

$a_3 - a_2 = 2$

$a_2 - a_1 = 2$

Adding, we get $a_n - a_1 = 2+2+\cdots+2$ $(n-1)$ times

$\Rightarrow a_n - 2 = 2(n-1)$ $\quad$ [$\because a_1 = 2$]

$a_n = 2(n-1)+2 = 2n-2+2 = 2n$, $n \geq 1$

$\therefore \boxed{a_n = 2n}$ is the explicit form of a_n.

We prove this by mathematical induction.

Let $P(n): a_n = 2n$, $n \geq 1 \rightarrow (1)$

T.P $P(n)$ IS TRUE for $n \geq 1$

STEP 1 : BASIS STEP : TO VERIFY $P(1)$ IS TRUE.

Put $n=1$, $P(1) = a_1 = 2(1) \Rightarrow 2=2$ IS TRUE.

$\therefore P(1)$ IS TRUE.

STEP 2 : INDUCTIVE STEP : ASSUME $P(K)$ IS TRUE.

Put $n=K$ in (1), $a_K = 2K$, $K \geq 1$.

STEP 3 : T.P $P(K+1)$ IS TRUE.

We T.P $P(K+1) : a_{K+1} = 2(K+1)$, $K \geq 1$

$P(K+1) = a_{K+1} = a_K + 2$ $\quad$ [Recurrence formula $a_n = a_{n-1}+2$]

$= 2K+2$.

$= 2(K+1)$

$\therefore P(K+1)$ IS TRUE

Thus $P(K)$ IS TRUE $\Rightarrow P(K+1)$ IS TRUE.

Hence by induction $P(n)$ IS TRUE $\forall n \geq 1$.

H.W

(1) Solve the recurrence relation defined by $S_0 = 100$ & $S_K = (1.08) S_{K-1}$ for $K \geq 1$ [Hint: $S_K = (1.08)^K 100$].

(2) Solve the recurrence relation $T(K) = 2T(K-1)$ with the initial condition $T(0) = 1$.

METHOD 2: CHARACTERISTIC ROOTS METHOD

LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONST COEFFICIENTS	NON-HOMOGENEOUS LINEAR RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS
RECURRENCE RELATION: $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ where $c_1, c_2, \ldots, c_k$ are real no. & $c_k \neq 0$. order / degree: $n-(n-k) = k$	RECURRENCE RELATION: $c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} = f(x)$ where $c_0, c_1, \ldots, c_k$ are constant with $c_0 \neq 0$, $c_k \neq 0$.

CHARACTERISTIC EQUATION: $c_0r^n + c_1r^{n-1} + \cdots + c_n = 0$

CASE	GENERAL SOLUTION OF RECURRENCE RELATn
① r_1 & r_2 are real & distinct	$a_n^{(h)} = Ar_1^n + Br_2^n$, A & B are constant
② r_1 & r_2 are real & Equal	$a_n^{(h)} = (A+Bn)r^n$, A & B arbitrary constants
③ r_1 & r_2 are Complex	$a_n^{(h)} = r^n[A\cos n\theta + B\sin n\theta]$ where $r = \sqrt{\alpha^2+\beta^2}$, $\tan\theta = \frac{\beta}{\alpha}$

RECURRENCE RELATION	ORDER	TYPE
① $c(k) - 5c(k-1) + 6c(k-2) = 2$	$k-(k-2)=2$	Non-Homogeneous
② $s(k) - 4s(k-1) - 11s(k-2) + 30s(k-3) = 4^k$	$k-(k-3)=3$	Non-Homogeneous
③ $f_n = f_{n-1} + f_{n-2}$	$n-(n-2)=2$	Homogeneous
④ $H_n = 2H_{n-1} + 1$	1	Non-Homogeneous

PROBLEMS TO SOLVE HOMOGENEOUS RECURRENCE RELATION

① Solve $s(k) + 5s(k-1) = 0$, $s(0) = 6$

Sol: Given: Homogeneous Recurrence relation:

$s(k) + 5s(k-1) = 0$, $s(0) = 6$

Order: $k-(k-1) = k-k+1 = 1$.

∴ Given is a 1st order Homogeneous Recurrence Relatn.

characteristic eqn: $r + 5 = 0 \Rightarrow \boxed{r=-5}$ → REAL & DISTINCT.

General homogeneous Sol: Real & distinct: Ar^n

$\boxed{s(k) = A\cdot 5^k}$ → ①

To find constants: Initial Condition Gn $s(0) = 6$.

For $k=0$ in ①, $s(0) = A\cdot 5^0 \Rightarrow \boxed{A=6}$

SOLUTION: $s(k) = 6\cdot 5^k$.

② Solve $S(K)+3S(K-1)-4S(K-2)=0$, $S(0)=3$, $S(1)=-2$

Sol: Given : Recurrence Relation is homogeneous of order 2.

Characteristic Eqn : $r^2+3r-4=0$

$r^2+4r-r-4=0$

$r(r+4)-r(r+4)=0$

$(r+4)(r-1)=0$

$\therefore \boxed{r=-4,1}$ - Real & Distinct

General Solution : Real & Distinct : $a_n = Ar_1^n + Br_2^n$

$$\boxed{S(K) = A(-4)^K + B(1)^K}$$

To find the Constants : Given condition : $S(0)=3$, $S(1)=-2$

→ $K=0$, $S(0) = A(-4)^0 + B(1)^0$

$\boxed{3 = A+B} \longrightarrow$ ①

→ $K=1$, $S(1) = A(-4)^1 + B(1)^1$

$\boxed{-2 = -4A+B} \rightarrow$ ②

Solving ① & ② we get $A=1$ & $B=2$.

Solution : $\boxed{S(K) = 1(-4)^K + 2(1)^K}$

③ Solve $U_{n+3}-6U_{n+2}+11U_{n+1}-6U_n=0$, $U_0=2$, $U_1=0$, $U_2=-2$

Sol: Given : Recurrence Relation is homogeneous of order 3

characteristic Eqn : $r^3-6r^2+11r-6=0$

Put $r=1 \Rightarrow 1-6+11-6=0$ $\therefore$ $\underline{\underline{r=1}}$ is a root.

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$r^2-5r+6=0$

$r^2-3r-2r+6=0$

$r(r-3)-2(r-3)=0$

$(r-2)(r-3)=0$

$r=2,3$

$\therefore$ $r=1,2,3$ are Real & Distinct.

General Solution of homogenous eqn } Real & Distinct : $U_n^{(h)} = Ar_1^n + Br_2^n + Cr_3^n$

$$\therefore \boxed{U(n) = A(1)^n + B(2)^n + C(3)^n}$$

TO FIND CONSTANT : Gn Condition : $U(0)=2$, $U(1)=0$, $U(2)=-2$.

$n=0$, $U(0)=A+B+C \Rightarrow A+B+C=2 \rightarrow$ ①

$n=1$, $U(1) = A(1)^1+B(2)^1+C(3)^1 \Rightarrow A+2B+3C=0 \rightarrow$ ②

$n=2$, $U(2) = A(1)^2+B(2)^2+C(3)^2 \Rightarrow A+4B+9C=-2 \rightarrow$ ③

Solving ①, ②, ③ using calculator $A=5$, $B=-4$, $C=1$.

Solution : $\boxed{U(n) = 5(1)^n + (-4)2^n + 1(3)^n}$

(4) Solve the recurrence relation for the fibonacci sequence 1, 1, 2, 3, 5, 8, 13.

Sol: Gn: Fibonacci sequence : $F_n = F_{n-1} + F_{n-2}$; $F_1 = 1$, $F_2 = 1$ is a homogeneous equation of order 2

characteristic eqn : $r^2 - r - 1 = 0$ $a=1, b=-1, c=-1$

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)-1}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\boxed{r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}} \rightarrow \text{Real \& distinct}$$

General solution : Real & distinct : $F(n) = A r_1^n + B r_2^n$

$$F(n) = A\left[\frac{1+\sqrt{5}}{2}\right]^n + B\left[\frac{1-\sqrt{5}}{2}\right]^n$$

To find constant: $F_1 = 1$, $F_2 = 1$.

$n=1$, $F(1) = A\left[\frac{1+\sqrt{5}}{2}\right]^1 + \left[\frac{1-\sqrt{5}}{2}\right]$

$$\boxed{1 = \frac{1}{2}[A+B] + \frac{\sqrt{5}}{2}[A-B]} \rightarrow (1)$$

$n=2$, $F(2) = A\left[\frac{1+\sqrt{5}}{2}\right]^2 + B\left[\frac{1-\sqrt{5}}{2}\right]^2$

$$= A\left[\frac{1+5+2\sqrt{5}}{4}\right] + B\left[\frac{1+5-2\sqrt{5}}{4}\right]$$

$$= A\left[\frac{3+\sqrt{5}}{2}\right] + B\left[\frac{3-\sqrt{5}}{2}\right]$$

$$\boxed{1 = \frac{3}{2}[A+B] + \frac{\sqrt{5}}{2}[A-B]} \rightarrow (2)$$

(2) - (1) $\Rightarrow 0 = A+B \Rightarrow B = -A$

Sub B in (1) ; $1 = \frac{1}{2}[A-A] + \frac{\sqrt{5}}{2}[A-(-A)] \Rightarrow \boxed{A = \frac{1}{\sqrt{5}}}$

$\therefore \boxed{B = \frac{-1}{\sqrt{5}}}$

Solution : $F(n) = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n + \left(\frac{-1}{\sqrt{5}}\right)\left[\frac{1-\sqrt{5}}{2}\right]^n$, $n \geq 1$

(5) Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$, $n \geq 2$ & $a_0 = 1$, $a_1 = 2$.

Sol: Given : $a_n - 2a_{n-1} + 2a_{n-2} = 0$ is a homogeneous recurrence relation of order 2.

Characteristic equation : $r^2 - 2r + 2 = 0$, $a=1, b=-2; c=2$.

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{2 \pm i2}{2}.$$

$r = \frac{2 \pm 2i}{2} \Rightarrow \boxed{r = 1+i} = \alpha + i\beta \rightarrow$ Complex Root.

$\therefore$ Roots are Complex with $\alpha = 1$, $\beta = 1$.

GENERAL SOLUTION : COMPLEX ROOT : $a_n^{(h)} = r^n[A\cos n\theta + B\sin n\theta]$ (25)

where $r = \sqrt{\alpha^2+\beta^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}\frac{\beta}{\alpha} = \tan^{-1}\frac{1}{1}$

$\theta = \tan^{-1}(1) = \pi/4$

$\therefore \boxed{\theta = \pi/4}$

$\therefore a_n = (\sqrt{2})^n\left[A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right]$

TO FIND CONSTANT : Given $a_0=1$, $a_1=2$

$n=0$, $a_0 = (\sqrt{2})^0[A\cos 0 + B\sin 0]$

$\boxed{1 = A}$

$n=1$, $a_1 = (\sqrt{2})^1\left[A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}\right]$.

$2 = (\sqrt{2})\left[\frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}}\right] = \frac{\sqrt{2}}{\sqrt{2}}[A+B]$.

$A+B=2 \Rightarrow 1+B=2 \Rightarrow B=2-1 \Rightarrow \boxed{B=1}$

SOLUTION : $a_n = (\sqrt{2})^n\left[(1)\cos\frac{n\pi}{4} + (1)\sin\frac{n\pi}{4}\right]$

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(6) Solve $a_n = 6a_{n-1} - 9a_{n-2}$, $n \geq 2$, $a_0=2$, $a_1=3$

Sol: Given : $a_n - 6a_{n-1} + 9a_{n-2} = 0$ is a homogenous Recurrence relation with order 2.

Characteristic Eqn : $r^2 - 6r + 9 = 0$.

$r^2 - 3r - 3r + 9 = 0$

$r(r-3) - 3(r-3) = 0$

$(r-3)(r-3) = 0$

$\boxed{r = 3, 3} \Rightarrow$ REAL AND EQUAL.

9 → −3, −3

$\therefore$ Roots are real & equal.

General Solution : Real & equal : $a_n^{(h)} = (A+Bn)r^n$.

$\boxed{a_n^{(h)} = (A+Bn)3^n}$

To find Constants : Given Condition $a_0=2$, $a_1=3$.

$n=0$, $a(0) = [A+B(0)]3^0 \Rightarrow \boxed{A=2}$

$n=1$, $a(1) = [A+B(1)]3^1 \Rightarrow 3 = [2+B]3 \Rightarrow \boxed{B=-1}$

Solution : $a_n^{(h)} = [2+(-1)n]3^n$, $n \geq 0$

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(7) Solve $S(k) + 6S(k-1) + 12S(k-2) + 8S(k-3) = 0$.

Sol: Gn : Recurrence Relation is a homogeneous relation with order 3.

Characteristic Equation: $r^3 + 6r^2 + 12r + 8 = 0$.

$r=-2 \Rightarrow -8+24-24+8 = 0$.

$\therefore r=-2$ is one of the root

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ & 0 & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$r^2+4r+4=0 \Rightarrow r^2+$

$r(r+2)+2(r+2)=0$

$(r+2)(r+2)=0$

$r=-2,-2$

$\therefore r=-2,-2,-2$ are Repeated root.

General solution: $S(K)=[A+BK+CK^2]\,r^K$.

$S(K)=[A+BK+CK^2](-2)^K$ is the General Solution.

H.W: Solve

① $S(K)-7S(K-2)+6S(K-3)=0$, $S(0)=8$, $S(1)=6$, $S(2)=22$. [$r=1,2,3$]

② $f(n)=7f(n-1)-10f(n-2)$, $f(0)=4$ & $f(1)=17$ [$f(n)=2^n+3\cdot 5^n$]

③ Solve $S(K)-4S(K-1)-11S(K-2)+30S(K-3)=0$ given that $S(0)=0$, $S(1)=-35$, $S(2)=-85$. [$S(K)=2^K+4(-3)^K-5\cdot 5^K$].

④ Solve $D(K)-8D(K-1)+16D(K-2)=0$ where $D(2)=16$, $D(3)=80$. [$D(K)=\left(\frac{1}{2}+\frac{K}{4}\right)4^K$]

⑤ Solve $U_{n+2}-3U_{n+1}-4U_n=0$, $U_0=1$, $U_1=3$ [$U_n=\frac{4^{n+1}+(-1)^n}{5}$]

⑥ Solve $a_n=6a_{n-1}-11a_{n-2}+6a_{n-3}$, $a_0=2$, $a_1=5$, $a_2=15$. [$a_n=1\cdot 2^n+2\cdot 3^n$]

⑦ Solve $a_n+6a_{n-1}+12a_{n-2}+8a_{n-3}=0$, $a_0=1$, $a_1=2$, $a_2=2$ [$a_n=(1-4n+2n^2)(-2)^n$, $n\geq 0$.

⑧ Solve $a_n=3a_{n-1}+4a_{n-2}$, $n\geq 2$ & $a_0=2$, $a_1=5$. [$a_n=4^n-(-1)^n$, $n\geq 0$]

PROBLEMS ON NON-HOMOGENEOUS RECURRENCE RELATION

$C_0a_n+C_1a_{n-1}+C_2a_{n-2}+\cdots+C_ka_{n-k}=f(n)$, $C_0\neq 0$, $C_k\neq 0$.

$$a_n=a_n^{(h)}+a_n^{(p)}$$

S.No:	R.H.S $=f(n)$	TRIAL - FUNCTION.
①	b^n [If b is not a root of the characteristic equations]	Ab^n.
②	b^n [If b is a root of the C.E with multiplicity s]	An^sb^n
③	$P(n)$ [Polynomial of degree m]	$A_0+A_1n+A_2n^2+\cdots+A_mn^m$.
④	$c^nP(n)$ [If c is not a root of C.E]	$c^n[A_0+A_1n+A_2n^2+\cdots+A_mn^m]$
⑤	$c^nP(n)$ [If c is a root of C.E with multiplicity t]	$c^n[A_0+A_1n+A_2n^2+\cdots+A_mn^m]n^t$
⑥	c [Constant]	A.

1. (a) Solve $S(K)+5S(K-1)=9$, $S(0)=6$

(b) Solve $S(K)-5S(K-1)+6S(K-2)=2$ with $S(0)=1$, $S(1)=-1$

(c) Solve $S(K)-S(K-1)-6S(K-2)=-30$ with $S(0)=20$, $S(1)=-5$.

Sol: (c) Given: $S(K)-S(K-1)-6S(K-2)=-30$, $S(0)=20$, $S(1)=-5$ is non-homogeneous recurrence relation with order 2.

To Find: $S(K)=S^{(h)}(K)+S^{(P)}(K)$.

→ To find: $S^{(h)}_{(K)}=?$

Consider Homogeneous relation: $S(K)-S(K-1)-6S(K-2)=0$.

Characteristic Equation: $r^2-r-6=0$

Roots: $r=-2,3$ ⇒ Real & distinct.

Homogeneous solution, $\boxed{S_K^{(h)}=A(-2)^n+B(3)^n}$.

→ To find $S^P(K)=?$

R.H.S = 9 [Constant]

Trial Solution: $\boxed{S(K)=C}$, $\boxed{S(K-1)=C}$, $\boxed{S(K-2)=C}$ → (1)

To find C: Sub (1) in $S(K)-S(K-1)-6S(K-2)=-30$.

$C-C-6C=-30$ ⇒ $\boxed{C=5}$

Particular Solution, $\boxed{S^{(P)}_{(K)}=5}$

∴ General Solution, $S(K)=S^{(h)}_{(K)}+S^{(P)}(K)$

$$\boxed{S(K)=A(-2)^n+B(3)^n+5}$$

Given: Initial Condition $S(0)=20$, $S(1)=-5$.

$K=0$ ⇒ $S(0)=A+B+5$, ⇒ $20-5=A+B$ ⇒ $\boxed{A+B=15}$ → (2)

$K=1$ ⇒ $S(1)=-2A+3B+5$ ⇒ $-5-5=-2A+3B$ ⇒ $\boxed{-2A+3B=-10}$ → (3)

Solving (2) & (3) by calculator $\boxed{A=11}$ and $\boxed{B=4}$

∴ $\boxed{S(K)=11(-2)^n+4(3)^n+5}$

2. Solve $S(K)-2S(K-1)+S(K-2)=2$, with $S(0)=25$, $S(1)=16$ → (1)

Sol: Given: Non-homogeneous recurrence relation with order 2.

To find: $S(K)=S^{(h)}K+S^P(K)$

→ To find: $S^{(h)}(K)=?$

Homogeneous relation: $S(K)-2S(K-1)+S(K-2)=0$.

Characteristic Equation: r^2-2r+1

Root: $r=1,1$ [Real & Equal].

General homogeneous Solution: $[A+BK](1)^K$.

→ To find: $S^P(K)=?$

R.H.S = 2 [Constant].

Final Solution : $s^P(K) = D$..

To find : D → $\boxed{S(K)=D}$, $S(K-1)=D$, $S(K-2)=D$ sub in ①

$D-2D+D=2 \Rightarrow 0=2$ is not possible.

→ $\boxed{S(K)=KD}$, $S(K-1)=(K-1)D$, $S(K-2)=(K-2)D$. sub in ①

$KD-2KD+2D+KD-2D=2 \Rightarrow 0=2$ is not possible

→ $\boxed{S(K)=K^2D}$, $S(K-1)=(K-1)^2D$, $S(K-2)=(K-2)^2D$

$K^2D-2(K^2+1-2K)D+(K^2+4-2K)D=2$

$K^2D-2K^2D-2D+4KD+K^2D+4D-2KD=2 \Rightarrow 2D=2 \Rightarrow \boxed{D=1}$

Particular Solution : $s^{(P)}(K)=K^2D \Rightarrow \boxed{s^{(P)}(K)=K^2}$

∴ General Solution : $S(K)=S^h(K)+S^P(K) \Rightarrow \boxed{S(K)=(A+BK)+K^2}$

Given initial Condition : $S(0)=25$, $S(1)=16$.

$K=0$, $S(0)=A \Rightarrow \boxed{A=25}$

$K=1$, $S(1)=(A+B)+1 \Rightarrow 16=25+B \Rightarrow \boxed{B=-1}$

∴ $\boxed{S(K)=25-10K+K^2}$

---

2) (a) Solve $G(K)-7G(K-1)+10G(K-2)=6+8K$, $G(0)=1$, $G(1)=2$

(b) Solve $a_n-3a_{n-1}=2n$, $a_1=3$.

(c) Solve $S(K)+5S(K-1)+6S(K-2)=3K^2-2K+1$

(d) Solve $a_n-5a_{n-1}+6a_{n-2}=8n^2$, $a_0=4$, $a_1=7$.

Sol.(d) Given : $a_n-5a_{n-1}+6a_{n-2}=8n^2$, → ① $a_0=4$, $a_1=7$.

is a non-homogeneous relation with order 2.

To find General solution $a(n)=a^h(n)+a^P(n)$

→ To find $a^h(n)=?$

Homogeneous relation : $a_n-5a_{n-1}+6a_{n-2}=0$

Characteristic equation : $r^2-5r+6=0$

Root : $r=3,2$ [Real & Distinct].

$6 \to -3, -2$

Homogeneous Solution $\boxed{a^h(n)=A(2)^n+B(3)^n}$

To find $a^P(n)$: R.H.S $=8n^2$ [polynomial of degree 2]

Trial Solution $a^P(n)=A_0+A_1n+A_2n^2$

To find : A_0, A_1, A_2 using ①

$a_n=A_0+A_1n+A_2n^2$, $a_{n-1}=A_0+A_1(n-1)+A_2(n-1)^2$

$a_{n-2}=A_0+A_1(n-2)+A_2(n-2)^2$

① $\Rightarrow a_n-5a_{n-1}+6a_{n-2}=8n^2$

$(A_0+A_1n+A_2n^2)-5(A_0+A_1(n-1)+A_2(n-1)^2]+6[A_0+A_1(n-2)+A_2(n-2)^2]=8n^2$

$(A_0+A_1n+A_2n^2)-5(A_0+A_1n-A_1+A_2(n^2+1-2n)]+6[A_0+A_1n-2A_1+A_2(n^2+4-4n)]=8n^2$ (27)

$(A_0+A_1n+A_2n^2)-5A_0-5A_1n+5A_1-5A_2n^2-5A_2+10nA_2+6A_0+6A_1n-12A_1+6A_2n^2+24A_2-24A_2n=8n^2$

$n^2[A_2-5A_2+6A_2]+n[A_1-5A_1+10A_2+6A_1-24A_2]+[A_0-5A_0+5A_1-5A_2+6A_0-12A_1+24A_2]=8n^2$

Comparing on both the sides.

$A_2-5A_2+6A_2=8 \Rightarrow 2A_2=8 \Rightarrow \boxed{A_2=4}$

$A_1-5A_1+10A_2+6A_1-24A_2=0 \Rightarrow 2A_1-14A_2=0 \Rightarrow 2A_1=14(4) \Rightarrow \boxed{A_1=28}$

$A_0-5A_0+5A_1-5A_2+6A_0-12A_1+24A_2=0 \Rightarrow 2A_0-7A_1+19A_2=0 \Rightarrow 2A_0=7(28)-19(4)$

$2A_0=120 \Rightarrow \boxed{A_0=60}$

$\therefore$ Particular Solution: $a^{(P)}(n)=60+28n+4n^2$

$\therefore$ General Solution of non-homogeneous eqn $\left.\right\}$ $\boxed{a_n=A\cdot 2^n+B\cdot 3^n+60+28n+4n^2}$

Given condition: $a_0=4$, $a_1=7$

$n=0$, $a_0=A\cdot 2^0+B\cdot 3^0+60+28(0)+4(0)^2 \Rightarrow 4=A+B+60 \Rightarrow \boxed{A+B=-54} \rightarrow (1)$

$n=1$, $a_1=A\cdot 2^1+B\cdot 3^1+60+28(1)+4(1)^2 \Rightarrow 7=2A+3B+92 \Rightarrow \boxed{2A+3B=-85} \rightarrow (2)$

Solving (1) & (2) by Calculator $\boxed{A=-83}$ & $\boxed{B=27}$

$\therefore$ solution: $a_n=-83\cdot 2^n+27\cdot 3^n+60+28n+4n^2$.

---

4 (a) Solve the relation $a_n-2a_{n-1}=2^n$, $a_0=2$

$[r=2$, $R.H.S=2^n$, $a^P(n)=An^s b^n=An^1 2^n$, $s=$ No. of Repeated roots$]$

$\underbrace{}_{\text{same}}$

(b) Solve the general solution of the relation $a_n-5a_{n-1}+6a_{n-2}=4^n$ $n\geq 2$.

$[r=2,3$, $R.H.S=4^n$, $a^P(n)=Ab^n]$

$\underbrace{}_{\text{Not same}}$

(c) Solve the recurren relation $a_{n+2}-6a_{n+1}+9a_n=3\cdot 2^n+7\cdot 3^n$ when $n\geq 0$ & $a_0=1$, $a_1=4$.

Sol: (c) Given: $a_{n+2}-6a_{n+1}+9a_n=3\cdot 2^n+7\cdot 3^n \rightarrow (1)$ is a non-homogeneous recurrence relation with order 2.

To find: General Solution: $a(n)=a_n^{(h)}+a_n^{(P)}$

$\rightarrow$ To FIND $a_n^{(h)}=?$

Homogeneous Eqn: $a_{n+2}-6a_{n+1}+9a_n=0$.

Characteristic Eqn: $r^2-6r+9=0 \Rightarrow r^2-3r-3r+9=0$

$(r-3)(r-3)=0 \Rightarrow \boxed{r=3,3}$

-6

$-3\ -3$

Roots [Real and Equal] : $r = 3, 3$. [Here s = no. of time 3 occurs = 2 times]

Homogeneous Solution: $\boxed{a_n^{(h)} = (A+Bn)3^n}$

To find $a_n^{(P)} = ?$ $\quad$ R.H.S $= 3 \cdot 2^n + 7 \cdot 3^n$.

Trial Solution : $a_n : C2^n + Dn^2 3^n$ $\quad$ [$r=3$, R.H.S $= 3^n \Rightarrow a_n^{(P)} = Dn^s b^n$ — same $= Dn^2 3^n$; $r \neq 3$, R.H.S $= 2^n \Rightarrow a_n^{(P)} = Cb^n = C2^n$]

To find C & D : $a_n = C2^n + Dn^2 3^n$

$$a_{n+1} = C2^{n+1} + D(n+1)^2 3^{n+1}$$

$$a_{n+2} = C\,2^{n+2} + D\,(n+2)^2\,3^{n+2}.$$

① $\Rightarrow a_{n+2} - 6a_{n+1} + 9a_n = 3 \cdot 2^n + 7 \cdot 3^n$

$$C2^{n+2} + D(n^2+4+4n)3^{n+2} - 6[C2^{n+1} + D(n^2+1+2n)3^{n+1}] + 9[C2^n + Dn^2 3^n] = 3 \cdot 2^n + 7 \cdot 3^n.$$

$$4C2^{n} + [9Dn^2 + 36D + 36Dn]3^n - 12C2^n - [18Dn^2 + 18 + 36n]3^n + 9[C2^n + Dn^2 3^n] = 3 \cdot 2^n + 7 \cdot 3^n,$$

Equating both the sides the coeff of 2^n & 3^n.

$$4C - 12C + 9C = 3 \Rightarrow \boxed{C=3}$$

$$9\cancel{D}n^2 + 36D + 36\cancel{D}n - 18\cancel{D}n^2 - 18 - 3\cancel{6}n + 9\cancel{D}n^2 = 7 \Rightarrow 18D = 7 \Rightarrow \boxed{D = 7/18}$$

$\therefore$ Particular Solution is $\boxed{a_n^{(P)} = 3 \cdot 2^n + \frac{7}{18} n^2 3^n.}$

General Solution of Non homogeneous eqn $\Big\}$ $\boxed{a_n = (A+Bn)3^n + 3 \cdot 2^n + \frac{7}{18} n^2 3^n}$.

Given Condition : $a_0 = 1$, $a_1 = 4$.

$n=0 \Rightarrow a_0 = (A+B(0))3^0 + 3 \cdot 2^0 + \frac{7}{8}(0)^2 3^0 \Rightarrow 1 = A+3 \Rightarrow \boxed{A=-2}$

$n=1 \Rightarrow a_1 = (A+B(1))3^1 + 3 \cdot 2^1 + \frac{7}{18}(1)^2 3^1 \Rightarrow 4 = (-2+B)3 + 6 + \frac{7}{6}.$

$$4 = -\cancel{6} + 3B + \cancel{6} + \frac{7}{6} \Rightarrow 3B = 4 - \frac{7}{6} = \frac{24-7}{6} = \frac{17}{6} \Rightarrow \boxed{B = \frac{17}{18}}$$

$\therefore$ Solution is $a_n = \left(-2 + \frac{17}{18}n\right)3^n + 3 \cdot 2^n + \frac{7}{18} n^2 \cdot 3^n$.

---

⑤ (*) Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + 4(n+1) \cdot 3^n$ where $a_0 = 2$, $a_1 = 3$.

Sol: Gn : $a_n - 6a_{n-1} + 9a_{n-2} = 4(n+1) \cdot 3^n \to$ ①, $a_0 = 2$, $a_1 = 3$ is a Non-Homogeneous recurrence relation with order 2.

To find : General Solution $a_n = a_n^{(h)} + a_n^{(P)}$

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To find: $a_n^{(h)} = ?$

Homogeneous relation: $a_n - 6a_{n-1} + 9a_{n-2} = 0$.

Characteristic equation: $r^2 - 6r + 9 = 0$

Roots [Real and equal]: $r = 3, 3$.

$$9 \to -3, -3.$$

General Homogeneous Solution: $a_n^{(h)} = [A + Bn]\, 3^n$.

To find: $a_n^{(p)} = ?$

$$R.H.S = 4(n+1) \cdot 3^n = (n+1) \cdot 4 \cdot 3^n$$

[R.H.S = (n+1) = Poly of deg 1 $= C + Dn$

R.H.S $= 3^n$ $(r=3)$ = $En^2 3^n$ Lsame]

Final Solution: $a_n^{(p)} = [C + Dn] \cdot [E n^2 3^n]$

$$= [EC + EDn] \cdot n^2 3^n$$

$$\boxed{a_n^{(p)} = [A_0 + A_1 n] \cdot n^2 3^n}$$ where $A_0 = EC$, $A_1 = ED$.

To find: A_0 & A_1

$a_n = n^2 3^n (A_0 + A_1 n)$; $a_{n-1} = (n-1)^2 3^{n-1} (A_0 + A_1(n-1))$

$a_{n-2} = (n-2)^2 3^{n-2} (A_0 + A_1(n-2))$

Sub in (1) $\Rightarrow$ $a_n - 6a_{n-1} + 9a_{n-2} = 4(n+1) 3^n$.

$n^2 3^n [A_0 + A_1 n] - 6[(n^2+1-2n) 3^{n-1} (A_0 + A_1 n - A_1)] +$

$9[(n^2+4-4n) \cdot 3^{n-2} (A_0 + A_1 n - 2A_1)] = 4(n+1) \cdot 3^n$.

$\cancel{3^n}[A_0 n^2 + A_1 n^3 - [(2n^2 + 2 - 4n)(A_0 + A_1 n - A_1)] +$

$[(n^2+4-4n)(A_0 + A_1 n - 2A_1)] = 4(n+1) \cdot \cancel{3^n}$.

$[A_0 n^2 + A_1 n^3 - (2n^2 A_0 + 2n^3 A_1 - 2n^2 A_1 + 2A_0 + 2A_1 n - 2A_1 - 4A_0 n$

$- 4A_1 n^2 + 4nA_1) + (A_0 n^2 + A_1 n^3 - 2A_1 n^2 + 4A_0 + 4A_1 n - 8A_1$

$- 4nA_0 - 4A_1 n^2 + 8A_1 n)] = 4(n+1) = 4n + 4$.

$n^3[\cancel{A_1} - 2\cancel{A_1} + \cancel{A_1}] + n^2[\cancel{A_0} - 2\cancel{A_0} + 2\cancel{A_1} + 4\cancel{A_1} + \cancel{A_0} - 2\cancel{A_1} - 4\cancel{A_1}]$

$+ n[-2A_1 + 4\cancel{A_0} - 4\cancel{A_1} + 4\cancel{A_1} - 4\cancel{A_0} + 8A_1] + [-2A_0 + 2A_1 + 4A_0 - 8A_1]$

$= 4n + 4$.

$n^3(0) + n^2(0) + n(6A_1) + [2A_0 - 6A_1] = 4n + 4$.

Equating on both sides.

$6A_1 = 4 \Rightarrow A_1 = \frac{4}{6} = \frac{2}{3} \Rightarrow \boxed{A_1 = \frac{2}{3}}$

$2A_0 - 6A_1 = 4 \Rightarrow 2A_0 = 4 + 6\left(\frac{2}{3}\right) \Rightarrow 2A_0 = 8 \Rightarrow \boxed{A_0 = 4}$

Particular Solution: $\boxed{a_n^{(p)} = [4 + \frac{2}{3} n]\, n^2 3^n}$

General solution of Non Homogeneous Solt $\}$ $a_n = (A + Bn) 3^n + \left(4 + \frac{2n}{2}\right) n^2 \cdot 3^n$

Given Initial Condition : $a_0=2$, $a_1=3$

$n=0 \Rightarrow a_0=A \Rightarrow \boxed{A=2}$

$n=1 \Rightarrow a_1=(A+B)3+\left[4+\frac{2}{3}\right]1\cdot3 = 6+3B+14$

$3 = 20+3B$

$\Rightarrow \boxed{B=-\frac{17}{3}}$

$\therefore$ The Solution of Non-Homogeneous Eqn,

$$a_n = \left[2+\left(-\frac{17}{3}\right)n\right]3^n + \left(4+\frac{2n}{3}\right)n^2\cdot 3^n.$$

METHOD 4: THE GENERATING FUNCTION (***) 8

The generating function of the sequence $a_0, a_1, a_2, \ldots, a_n$ of real numbers is the infinite series:

$$\boxed{G(x) = a_0 + a_1x + a_2x^2 + \cdots + \cdots = \sum_{n=0}^{\infty} a_nx^n}$$

SOME USEFUL EXPANSION:

1. $\sum_{n=0}^{\infty} 1^n x^n = 1+x+x^2+\cdots = (1-x)^{-1}$
2. $\sum_{n=0}^{\infty} (-1)^n x^n = 1-x+x^2-\cdots = (1+x)^{-1}$
3. $\sum_{n=0}^{\infty} a^n x^n = 1+ax+(ax)^2+(ax)^3+\cdots = (1-ax)^{-1}$
4. $\sum_{n=0}^{\infty} (-a)^n x^n = 1-(ax)+(ax)^2-(ax)^3+\cdots = (1+ax)^{-1}$
5. $\sum_{n=0}^{\infty} (n+1)x^n = 1+2x+3x^2+4x^3+\cdots = \frac{1}{(1-x)^2} = (1-x)^{-2}$
6. $\sum_{n=0}^{\infty} n\cdot x^n = x+2x^2+3x^3+4x^4+\cdots = \frac{x}{(1-x)^2}$
7. $\sum_{n=0}^{\infty} n^2x^n = x+2^2x^2+3^2x^3+4^2x^4+\cdots = \frac{x(x+1)}{(1-x)^3}$
8. $\sum_{n=0}^{\infty} (n+2)^2x^n = 2^2+3^2x+4^2x^2+5^2x^3+\cdots = \frac{x+1}{(1-x)^3}$
9. $\sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = e^x$
10. $\sum_{n=0}^{\infty} n(n+1)x^n = 0+1\cdot2x^1+2\cdot3x^2+3\cdot4x^3+\cdots = \frac{2x}{(1-x)^3}$.

PROBLEMS UNDER GENERATING FUNCTION.

① Solve $a_n-9a_{n-1}+20a_{n-2}=0$, $a_0=-3$, $a_1=-10$ using generating function.

Sol: STEP 1: Given $a_n-9a_{n-1}+20a_{n-2}=0 \to (1)$, $a_0=-3$, $a_1=-10$

STEP 2: Let $G(x) = \sum_{n=0}^{\infty} a_nx^n = a_0+a_1x+a_2x^2+\cdots$

STEP 3: Multiply by x^n in (1) and $\sum_{n=2}^{\infty}$ (*)

$$\sum_{n=2}^{\infty} a_n x^n - 9\sum_{n=2}^{\infty} a_{n-1}x^n + 20\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - 9x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 20x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0.$$

STEP 4: TO FIND $G(x)$

$$[a_2x^2+a_3x^3+\dots] - 9x[a_1x^1+a_2x^2+\dots] + 20x^2[a_0+a_1x+a_2x^2\dots] = 0$$

$$[(a_0+a_1x+a_2x^2+\dots) - a_0 - a_1x] - 9x[(a_0+a_1x+a_2x^2+\dots) - a_0] + 20x^2[G(x)] = 0$$

$$[G(x) - a_0 - a_1x] - 9x[G(x) - a_0] + 20x^2[G(x)] = 0.$$

$\because a_0 = -3$ and $a_1 = -10$.

$$[G(x) + 3 + 10x] - 9x[G(x) + 3] + 20x^2 G(x) = 0.$$

$$G(x)[1 - 9x + 20x^2] = -3 - 10x + 27x = -3 + 17x.$$

$$\boxed{G(x) = \frac{17x-3}{20x^2-9x+1}}$$

STEP 5: USING PARTIAL FRACTION

Consider $\dfrac{17x-3}{20x^2-9+1} = \dfrac{17x-3}{20x^2-5x-4x+1} = \dfrac{17x-3}{5x(4x-1)-(4x-1)}$

$$= \frac{17x-3}{(4x-1)(5x-1)} = \frac{17x-3}{(1-4x)(1-5x)}$$

$$G(x) = \frac{17x-3}{(1-4x)(1-5x)} = \frac{A}{(1-5x)} + \frac{B}{(1-4x)}$$

$$17x-3 = A(1-4x) + B(1-5x).$$

Put $x = \frac{1}{4} \Rightarrow \boxed{A=5}$

Put $x = \frac{1}{5} \Rightarrow \boxed{B=2}$

$$\therefore G(x) = \frac{5}{1-5x} + \frac{2}{1-4x}.$$

STEP 6: $\sum_{n=0}^{\infty} a_n x^n = 5\sum_{n=0}^{\infty} 5^n x^n + 2\sum_{n=0}^{\infty} 4^n x^n$

– Equating Coefficient of x^n. $\left[\sum_{n=0}^{\infty} a^n x^n = (1-ax)^{-1}\right]$

$$\boxed{a_n = 5\cdot 5^n + 2\cdot 4^n}$$

② Solve $y_{n+2} - 5y_{n+1} + 6y_n = 0$, $y_0 = 1$, $y_1 = 1$ by generating function.

Sol: STEP1: $a_{n+2} - 5a_{n+1} + 6a_n = 0$, $a_0 = 1$, $a_1 = 1$.

STEP2: $G(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + \cancel{a}x + a_2x^2 + a_3x^3 + \cdots$

STEP3: Multiply by x^n & $\sum_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 5\sum_{n=0}^{\infty} a_{n+1}x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - \frac{5}{x}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

STEP4: TO FIND $G(x)$

$$\frac{1}{x^2}[a_2x^2 + a_3x^3 + \cdots] - \frac{5}{x}[a_1x + a_2x^2 + \cdots] + 6[a_0 + a_1x + a_2x^2] = 0.$$

$$\frac{1}{x^2}\left[(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) - a_0 - a_1x\right] - \frac{5}{x}\left[(a_0 + a_1x + a_2x^2 + \cdots) - a_0\right]$$

$$+ 6[a_0 + a_1x + a_2x^2 + \cdots] = 0$$

$$\frac{1}{x^2}[G(x) - a_0 - a_1x] - \frac{5}{x}[G(x) - a_0] + 6[G(x)] = 0.$$

$$[G(x) - a_0 - a_1x] - 5x[G(x) - a_0] + 6x^2 G(x) = 0$$

$$G(x)[1 - 5x + 6x^2] = a_0 + a_1x - 5xa_0 = 1 + x - 5x = 1 - 4x$$

$$\therefore \boxed{G(x) = \frac{1-4x}{6x^2 - 5x + 1}} = \frac{1-4x}{(1-2x)(1-3x)}$$

6
-3 -2

STEP 4: USE PARTIAL FRACTION

Consider $\frac{1-4x}{(1-2x)(1-3x)} = \frac{A}{(1-2x)} + \frac{B}{(1-3x)}$

$$1 - 4x = A(1-3x) + B(1-2x)$$

Put $x = \frac{1}{3} \Rightarrow \boxed{B = -1}$

$x = \frac{1}{2} \Rightarrow \boxed{A = 2}$

$$\therefore G(x) = \frac{2}{(1-2x)} + \frac{-1}{(1-3x)}$$

STEP 5:- $\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 2^n x^n - 1\sum_{n=0}^{\infty} 3^n x^n.$

Equating the coefficients of x^n.

$$\boxed{a_n = 2 \cdot 2^n - 1 \cdot 3^n}$$

$$\left[\because \sum_{n=0}^{\infty} a^n x^n = (1-ax)^{-1}\right]$$

③ Solve $a_n = 3a_{n-1}+2$ with $a_0=1$ using generating function

Sol: STEP 1: $a_n - 3a_{n-1} = 2$, $a_0 = 1$ (30)

STEP 2: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

STEP 3: Multiply by x^n and $\sum_{n=1}^{\infty}$

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = 2\sum_{n=1}^{\infty} x^n.$$

$$\sum_{n=1}^{\infty} a_n x^n - 3x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 2\sum_{n=1}^{\infty} x^n$$

STEP 4: TO FIND $G(x)$.

$$[a_1x^1 + a_2x^2 + \dots] - 3x[a_0 + a_1x + a_2x^2 + \dots] = 2[x + x^2 + \dots]$$

$$[a_0 + a_1x + a_2x^2 + \dots - a_0] - 3x[a_0 + a_1x + \dots] = 2x[1 + x + \dots].$$

$$G(x) - a_0 - 3xG(x) = 2x(1-x)^{-1}$$

$$G(x)[1-3x] = \frac{2x}{(1-x)} + 1 = \frac{2x+1-x}{1-x} = \frac{x+1}{1-x}.$$

$$\therefore G(x) = \frac{x+1}{(1-3x)(1-x)}$$

Consider $\frac{x+1}{(1-3x)(1-x)} = \frac{A}{(1-3x)} + \frac{B}{(1-x)}.$

Put $x = 1/3 \Rightarrow \boxed{A=3}$

$x = 1 \Rightarrow \boxed{B=-1}.$

$$\therefore \boxed{G(x) = \frac{3}{(1-3x)} - \frac{1}{(1-x)}}.$$

STEP 5: $\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} 1^n x^n.$

Equating coeff of a^n we get -

$$\boxed{a_n = 3 \cdot 3^n - 1^n}$$

---

④ Solve the recurrence relation $s(n+1) - 2s(n) = 4^n$, $s(0)=1$

Sol: STEP 1: $a_{n+1} - 2a_n = 4^n$, $a_0 = 1$, $n \geq 0$.

STEP 2: $G(x) = \sum_{n=0}^{\infty} a_n x^n.$

STEP 3: Multiply x^n & $\sum_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty} a_{n+1} x^n - 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 4^n x^n. \qquad \left[\because \sum_{n=0}^{\infty} a^n x^n = (1-ax)^{-1}\right].$$

$$\frac{1}{x}\sum_{n=0}^{\infty} a_{n+1} x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^n = (1-4x)^{-1}$$

STEP 4: TO FIND $G(x)$.

$$\frac{1}{x}[a_1x + a_2x^2 + \dots] - 2[a_0 + a_1x + a_2x^2 + \dots] = \frac{1}{(1-4x)}$$

$\frac{1}{x}[a_0+a_1x+a_2x^2+\cdots-a_0]-2[G(x)=\frac{1}{(1-4x)}$

$[G(x)-a_0]-2x\,G(x)=\frac{x}{1-4x}$

$G(x)[1-2x]=\frac{x}{1-4x}+1=\frac{x+1-4x}{1-4x}=\frac{1-3x}{1-4x}$

$$\boxed{G(x)=\frac{1-3x}{(1-2x)(1-4x)}}$$

Consider $\frac{1-3x}{(1-2x)(1-4x)}=\frac{A}{(1-2x)}+\frac{B}{(1-4x)}$

$1-3x=A(1-4x)+B(1-2x)$

put $x=\frac{1}{4}\Rightarrow\boxed{B=\frac{1}{2}}$

$x=\frac{1}{2}\Rightarrow\boxed{A=\frac{1}{2}}$

$\therefore G(x)=\frac{\frac{1}{2}}{(1-2x)}+\frac{\frac{1}{2}}{(1-4x)}$

STEP 5: $\sum_{n=0}^{\infty}a_nx^n=\frac{1}{2}\sum_{n=0}^{\infty}2^nx^n+\frac{1}{2}\sum_{n=0}^{\infty}4^nx^n$.

Equating coeff of x^n.

$$\boxed{a_n=\frac{1}{2}\cdot2^n+\frac{1}{2}\cdot4^n}$$

---

⑥ Solve $a_{n+2}-2a_{n+1}+a_n=2^n$, $a_0=2$, $a_1=1$ by G.F.M.

Sol: STEP 1: $a_{n+2}-2a_{n+1}+a_n=2^n$, $a_0=2$, $a_1=1$.

STEP 2: Let $G(x)=\sum_{n=0}^{\infty}a_nx^n$

STEP 3: Multiply by x and $\sum_{n=0}^{\infty}$

$\sum_{n=0}^{\infty}a_{n+2}x^n-2\sum_{n=0}^{\infty}a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}2^nx^n$.

$\frac{1}{x^2}\sum_{n=0}^{\infty}a_{n+2}x^{n+2}-\frac{2}{x}\sum_{n=0}^{\infty}a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_nx^n=(1-2x)^{-1}$

STEP 3: TO FIND G(x)

$\frac{1}{x^2}[a_2x^2+a_3x^3+\cdots]-\frac{2}{x}[a_1x^1+a_2x^2+\cdots]+\sum_{n=0}^{\infty}a_nx^n=\frac{1}{(1-2x)}$.

$\frac{1}{x^2}[a_0+a_1x+a_2x^2+\cdots-a_0-a_1x]-\frac{2}{x}[a_0+a_1x+a_2x^2+\cdots-a_0]$

$+G(x)=\frac{1}{1-2x}$.

$\frac{1}{x^2}[G(x)-2-x]-\frac{2}{x}[G(x)-2]+G(x)=\frac{1}{1-2x}$

$[G(x)-2-x]-2x[G(x)-2]+x^2G(x)=\frac{x^2}{1-2x}$.

$G(x)[1-2x+x^2]=\frac{x^2}{1-2x}+2+x-4x.=\frac{1}{1-2x}+2-3x$

$$= \frac{x^2 + 2(1-2x) - 3x(1-2x)}{1-2x} = \frac{x^2+2-4x-3x+6x^2}{1-2x}$$

$$= \frac{7x^2-7x+2}{(1-2x)}$$

$$G(x) = \frac{7x^2-7x+2}{(1-2x)(x^2-2x+1)} = \frac{7x^2-7x+2}{(1-2x)(x-1)(x-1)} = \frac{7x^2-7x+2}{(1-2x)(x-1)^2}$$

Consider, $\frac{7x^2-7x+2}{(1-2x)(1-x)^2} = \frac{A}{(1-2x)} + \frac{B}{(1-x)} + \frac{C}{(x-1)^2}$

$7x^2-7x+2 = A(1-x)^2 + B(1-2x)(1-x) + C(1-2x)$.

Put $x=1 \Rightarrow \boxed{C=-2}$

$x=\frac{1}{2} \Rightarrow \boxed{A=1}$

Equating constant coefficient : $2 = A+B+C = 1+B-2 \Rightarrow \boxed{B=3}$

$$\boxed{G(x) = \frac{1}{(1-2x)} + \frac{3}{(1-x)} + \frac{-2}{(1-x)^2}}$$

STEP 5: $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n + 3\sum_{n=0}^{\infty} 1^n x^n - 2\sum_{n=0}^{\infty} (n+1) x^n$.

Equating coeff of a^n.

$a_n = 2^n + 3\cdot 1^n - 2(n+1) = 2^n + 3 - 2n - 2 = 2^n - 2n + 1$

$$\boxed{a_n = 2^n - 2n + 1}$$

---

(6) Use the method of generating function to solve the recurrence relation $a_n = 4a_{n-1} + 3n\cdot 2^n$, $n \geq 1$, $a_0 = 4$.

Sol: STEP 1: $a_n - 4a_{n-1} = 3n\cdot 2^n$, $a_0 = 4$.

STEP 2: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

STEP 3: Multiply by x^n and $\sum_{n=1}^{\infty}$: $n \geq 1$

$\sum_{n=1}^{\infty} a_n x^n - 4\sum_{n=1}^{\infty} a_{n-1} x^n = 3\sum_{n=1}^{\infty} n\cdot 2^n x^n$. → [Should be in such way that when you substitute n we should have 1 as 1st term]

$\sum_{n=1}^{\infty} a_n x^n - 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 3\sum_{n=1}^{\infty} n\cdot(2x)^n$

STEP 4: TO FIND $G(x)$

$[a_1x^1 + a_2x^2 + \cdots] - 4x[a_0x^0 + a_1x + a_2x^2 + \cdots] = 3\times 2x \sum_{n=1}^{\infty} n(2x)^{n-1}$

$[a_0 + a_1x + a_2x^2 + \cdots - a_0] - 4x[a_0 + a_1x + a_2x^2 + \cdots] = 6x[1 + 2(2x) + 3(2x)^2 + 4(2x)^3 + \cdots]$

$[1 + 2x + 3x^2 + \cdots = (1-x)^{-2}]$

$[G(x) - a_0] - 4x[G(x)] = 6(x)(1-2x)^{-2}$. $[\because a_0 = 4]$

$$G(x)[1-4x] = \frac{6x}{(1-2x)^2} + 4.$$

$$G(x) = \frac{6x}{(1-4x)(1-2x)^2} + \frac{4}{(1-4x)}$$

Consider $\frac{6x}{(1-2x)^2(1-4x)} = \frac{A}{(1-4x)} + \frac{B}{(1-2x)} + \frac{C}{(1-2x)^2}$.

$$6x = A(1-2x)^2 + B(1-4x)(1-2x) + C(1-4x).$$

Put $x=\frac{1}{2} \Rightarrow 3 = -C \Rightarrow \boxed{C=-3}$

$x=\frac{1}{4} \Rightarrow \frac{6}{4} = A\left(1-\frac{2}{4}\right)^2 \Rightarrow \frac{3}{2} = A\left(\frac{1}{2}\right)^2 \Rightarrow \boxed{A=6}$

Equating the constant coeff : $0 = A+B+C \Rightarrow 0 = 6+B-3$.

$\Rightarrow \boxed{B=-3}$

$\therefore G(x) = \frac{6}{(1-4x)} + \frac{-3}{(1-2x)} + \frac{-3}{(1-2x)^2} + \frac{4}{(1-4x)}$

STEP 5 $\sum_{n=0}^{\infty} a_n x^n = 6\sum_{n=0}^{\infty} 4^n x^n + -3\sum_{n=0}^{\infty} (2)^n x^n - 3\sum_{n=0}^{\infty} (n+1)x^n + 4\sum_{n=0}^{\infty} 4^n x^n$.

Equating the coeff of x^n.

$$\boxed{a_n = 6\cdot 4^n - 3\cdot 2^n - 3\, 2^n(n+1) + 4\cdot 4^n.}$$

$\left[\because \sum_{n=0}^{\infty}(n+1)x^n = (1-x)^{-2} \quad \sum_{n=0}^{\infty} a^n x^n = (1-ax)^{-1}\right]$

⑦ Solve using generating function $a_n = 8a_{n-1} + 10^{n-1}$, $a_1 = 9$

Sol: Step 1 : $a_n - 8a_{n-1} = 10^{n-1}$.

STEP 2 : $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

STEP 3 : Multiply by x and $\sum_{n=1}^{\infty}$

$\sum_{n=1}^{\infty} a_n x^n - 8\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} 10^{n-1}x^n$:

$\sum_{n=1}^{\infty} a_n x^n - 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = \sum_{n=1}^{\infty} 10^{n-1}x^n$.

STEP 4 : TO FIND $G(x)$

$[a_1x^1 + a_2x^2 + \cdots] - 8x[a_0 + a_1x + a_2x^2 + \cdots] = x + 10x^2 + 10^2x^3 + \cdots$

$[a_0 + a_1x + a_2x^2 + \cdots - a_0] - 8x[a_0 + a_1x + \cdots] = x[1 + (10x) + (10x)^2 + \cdots]$.

$[G(x) - a_0] - 8x[G(x)] = x(1-10x)^{-1}$ $\left[\because \sum_{n=0}^{\infty} a^n x^n = (1-ax)^{-1}\right]$

$G(x)[1-8x] = \frac{x}{(1-10x)} + a_0 = \frac{x}{1-10x} + 1 = \frac{x + 1 - 10x}{1-10x} = \frac{1-9x}{1-10x}$

$\therefore G(x) = \frac{1-9x}{(1-8x)(1-10x)} = \frac{A}{1-8x} + \frac{B}{1-10x}$.

Given $a_1 = 9$
$a_0 = ?$
$a_1 = 8a_0 + 10^0$
$9 = 8a_0 + 1$
$8 = 8a_0$
$\Rightarrow \boxed{a_0 = 1}$

$\therefore 1 - 9x = A(1-10x) + B(1-8x)$.

Put $x = \frac{1}{8} \Rightarrow \boxed{A = 1/2}$ and $x = \frac{1}{10} \Rightarrow \boxed{B = \frac{1}{2}}$

$\therefore G(x) = \frac{\frac{1}{2}}{1-8x} + \frac{\frac{1}{2}}{1-10x}$

STEP : 5 $\sum a_n x^n = \frac{1}{2}\sum_{n=0}^{\infty} 8^n x^n + \frac{1}{2}\sum_{n=0}^{\infty} 10^n x^n$.

Equating coeff of x^n we get : $a_n = \frac{1}{2}\cdot 8^n + \frac{1}{2}\cdot 10^n$

(8) VERY VERY IMPORTANT UQ

Find the recurrence relation of fibonacci seq using generating function and solve it.

Sol: STEP 1: Fibbonacci Sequence $a_n - a_{n-1} - a_{n-2} = 0,\ a_0 = 1,\ a_1 = 1$

STEP 2: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

STEP 3: Multiply by x^n and $\sum_{n=2}^{\infty}$

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

STEP 4: TO FIND $G(x)$

$$[a_2x^2 + a_3x^3 + \dots] - x[a_1x + a_2x^2 + \dots] - x^2[a_0 + a_1x + a_2x^2 + \dots] = 0$$

$$[G(x) - a_0 - a_1x] - x[G(x) - a_0] - x^2 G(x) = 0. \quad [\because a_0 = 1, a_1 = 1]$$

$$G(x) - 1 - x - xG(x) + x - x^2 G(x) = 0$$

$$G(x)[1 - x - x^2] = 1 \qquad [\because a - bx - cx^2 = (1-ax)(1-bx)]$$

$$\boxed{G(x) = \frac{1}{1-x-x^2}}$$

Consider, $1 - x - x^2 = 0$, $x = \frac{-(-1) \pm \sqrt{1-4(-1)(1)}}{2(-1)} = \frac{1 \pm \sqrt{5}}{-2}$

$$\therefore (1-x-x^2) = \left[1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right]\left[1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right]$$

$$\therefore G(x) = \frac{1}{(1-x-x^2)} = \frac{1}{\left[1-\left(\frac{1+\sqrt{5}}{2}\right)x\right]\left[1-\left(\frac{1-\sqrt{5}}{2}\right)x\right]} = \frac{A}{\left(1-\left[\frac{1+\sqrt{5}}{2}\right]x\right)} + \frac{B}{\left(1-\left(\frac{1-\sqrt{5}}{2}\right)x\right)}$$

$$1 = A\left[1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right] + B\left[1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right]$$

Put $x = \frac{2}{1-\sqrt{5}} \Rightarrow \boxed{B = \frac{1-\sqrt{5}}{-2\sqrt{5}}}$ and Put $x = \frac{2}{1+\sqrt{5}} \Rightarrow \boxed{A = \frac{1+\sqrt{5}}{2\sqrt{5}}}$

$$\therefore G(x) = \frac{\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)}{\left(1-\left(\frac{1+\sqrt{5}}{2}\right)x\right)} + \frac{\left(\frac{1-\sqrt{5}}{-2\sqrt{5}}\right)}{\left(1-\left(\frac{1-\sqrt{5}}{2}\right)x\right)}$$

STEP 5: $$\sum_{n=0}^{\infty} a_n x^n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\sum_{n=0}^{\infty}\left(\frac{1+\sqrt{5}}{2}\right)^n x^n + \left(\frac{1-\sqrt{5}}{-2\sqrt{5}}\right)\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n x^n$$

$$\therefore a_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{-2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

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(9) Find the sequence having the expression $\frac{3-5x}{1-2x-3x^2}$ as a generating function.

Sol: Given $G(x) = \frac{3-5x}{(1-2x-3x^2)} = \frac{3-5x}{(1-3x)(1+x)}$

Now $\frac{3-5x}{(1-3x)(1+x)} = \frac{A}{(1-3x)} + \frac{B}{(1+x)}$

$3-5x = A(1+x) + B(1-3x)$.

Put $x=-1 \Rightarrow \boxed{B=2}$

$x = \frac{1}{3} \Rightarrow \boxed{A=1}$

$\therefore G(x) = \frac{1}{1-3x} + \frac{2}{1+x}$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 3^n . x^n + 2 \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\left[\because \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x} \qquad \sum_{n=0}^{\infty} a^n x^n = \frac{1}{1-ax} \right]$$

$a_n = 3^n + 2(-1)^n$

H.W

① Use method of generating function to solve the reccurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$, $n \geq 2$ gn that $a_0 = 0, a_1 = 8$

⑤ Identify the sequence having the expresion $\frac{5+2x}{1-4x^2}$ as a generating function.

DERANGEMENTS

A derangement is a permutation of object in a row that leaves no object in its original position

The no: of derangements of n object is

$$D_n = n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right] \text{ or } D_n = n!\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}$$

PROBLEMS

① There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box could be placed such that a ball does not go into a box of its own colour.

Sol: We can regard a ball going into a box of its colour as matching. Since no ball goes into a box of its own colour, it is a derangements of 4 objects.

The no: of ways is $D_n = 4!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right] = 9$ ways.

② A person writes letters to 6 friends and places them in address envelopes. In how many ways can he place the letters in the envelopes so that (i) all the letters are in the wrong envelopes (ii) Atleast two of them are in the wrong envelopes.

Sol (i) $D_6 = 6!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right] = 265$.

(ii) Atleast two of them are in wrong envelopes =

= Total no: of derangements − the correct placements

$= 6! - 1 = 719$.

Atleast 2 wrong

GENERAL FORMULAE FOR 1-1 & ONTO FUNCTION

Suppose $|A| = m$, $|B| = n$

No: of 1-1 function from A to B → $m > n$: n^m. ; $m \le n$: ${}^nP_m$

No: of onto function from A to B → $m \geq n$; $n^m - {}^nC_{n-1}(n-1)^m + {}^nC_{n-2}(n-2)^m + \cdots + (-1)^n$; $m < n$; 0

Find the no: of onto functions from a set with 6 elements to a set with 3 elements

Gn $m=6$, $n=3$, Here $m>n$.

∴ The no: of onto function $= 3^6 - {}^3C_2 2^6 + {}^3C_1 1^6 = 729 - 6 + 3 = 126$

How many ways 5 different jobs can be assigned to 4 different employees if every employee is assigned atleast one job → onto func

Here $m=5$, $n=4$ and $m>n$

The no: of onto function $= 4^5 - {}^4C_3 3^5 + {}^4C_2 2^5 - {}^4C_1 1^5 = 1024 - 972 + 192 - 4 = 240$

Q: 9 Find the recurrence relation of fibonacci seq using generating function and solve it. 12

Sol: Step 1: Recurrence relation $F_n = F_{n-1} + F_{n-2}$, $F_0 = 1$, $F_1 = 1$

ie) $a_n - a_{n-1} - a_{n-2} = 0 \rightarrow (1)$, $a_0 = 1$, $a_1 = 1$

Step 2: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Step 3: x'ly by x^n and $\sum_{n=2}^{\infty}$

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0 \qquad (1-ax)(1-bx)=0$$

$$[a_2x^2 + a_3x^3 + \cdots] - x[a_1x + a_2x^2 + \cdots] - x^2[a_0 + a_1x + a_2x^2 + \cdots] = 0$$

$$G(x) - a_0 - a_1x - x[G(x) - a_0] - x^2 G(x) = 0$$

$[\because a_0 = 1, a_1 = 1]$

$$G(x) - 1 - \cancel{x} - xG(x) + \cancel{x} - x^2 G(x) = 0$$

$$G(x)[1 - x - x^2] = 1$$

$$\therefore \boxed{G(x) = \frac{1}{(1-x-x^2)}}$$

Consider, $1 - x - x^2 = \qquad [a - bx - cx^2 = 0,\ a = +1,\ b = -1,\ c = 1]$

$$x = \frac{-(-1) \pm \sqrt{1 - 4(-1)(1)}}{2(-1)} = \frac{1 \pm \sqrt{5}}{2} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \boxed{x = a, b}$$

ie) $a - bx - cx^2 = (1-ax)(1-bx)$ (or) $= (x-a)(x-b)$

$$\therefore (1 - x - x^2) = \left(1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right)\left(1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right)$$

$$G(x) = \frac{1}{(1-x-x^2)} = \frac{1}{\left[1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right]\left[1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right]} = \frac{A}{\left[1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right]} + \frac{B}{\left[1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right]}$$

$$1 = A\left[1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right] + B\left[1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right]$$

Put $x = \left(\frac{2}{1-\sqrt{5}}\right) \Rightarrow B = \frac{1-\sqrt{5}}{-2\sqrt{5}}$

Put $x = \left(\frac{2}{1+\sqrt{5}}\right) \Rightarrow A = \frac{1+\sqrt{5}}{2\sqrt{5}}$

$$\therefore G(x) = \frac{\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)}{\left[1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right]} + \frac{\left(\frac{1-\sqrt{5}}{-2\sqrt{5}}\right)}{\left[1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right]} = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right)^{-1} - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\sum_{n=0}^{\infty}\left(\frac{1+\sqrt{5}}{2}\right)^n x^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n x^n$$

Equating the coeff of x^n we get

$$a_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$