unit - II

Introduction to proofs :-

Theorem Definition :-

Theorem is a statement that can be shown to be True.

Proof :-

A proof is a valid argument that establish the validity of the statement.

Types of proofs :-

5m
1. Direct proofs
2. proofs by contradiction } Indirect Proofs
3. proofs by contrapositive

10m
4. Induction proofs

1. Direct proofs :-

A Direct proofs of the form $p \Rightarrow q$ is logically a valid argument in which we start with the assumtion the 'p' is true and by using 'p' along with other axioms (or) Rules we show directly 'q' is also True. * Mathematically it is Denoted as $p \longrightarrow q$

Assume 'p' is True

Prove q is also True (by using some axioms)

Ex :- 1

Give a direct proof for the statement "Product of Two Odd integers is Odd"

sol :- $3 * 5 = 15$ similarly $7 * 5 = 35$

Let x, y be two odd integers.

By the rule of Mathematics odd number can be represented as "2n+1"

$$x \cdot y = \quad x = 2n+1$$
$$y = 2m+1$$

$$= (2n+1)(2m+1)$$
$$= 4nm + 2n + 2m + 1$$
$$= 2(\underbrace{2nm + n + m}_{a}) + 1$$

$$\boxed{x \cdot y = 2a+1}$$

This is an odd integer

∴ product of two odd integers is odd

Ex :- 2

Show that the square of an even number is also even

$4^2 = 16$ similary $6^2 = 36$

Let 'x' be a even number

then $x = 2n$

$$x^2 = (2n)^2$$
$$= 4n^2$$
$$= 2(\underbrace{2n^2}_{a})$$

$$\boxed{x^2 = 2a}$$

This is an even number

∴ square of even number is also even

Ex:- 3

Show that the some of two odd numbers is even

$3+7 = 10$ similarly $3+5 = 8$

Let x, y be two odd integers

$q : x+y$ is even

and $x = 2n+1$

$y = 2m+1$

$x+y = (2n+1) + (2m+1)$

$= 2n+2m+2$

$= 2(\underbrace{n+m+1}_{a})$

$\boxed{x+y = 2a}$

$x+y =$ even

$\therefore$ the sum of two odd integers is even

Ex:- 4

Show that the square of an odd integer is odd

$3^2 = 9$ similarly $5^2 = 25$

Let 'x' be a odd number

then $x = 2n+1$

$x^2 = (2n+1)^2$

$= 4n^2 + 1 + 4n$

$= 2(\underbrace{2n^2 + 2n}_{a}) + 1$

$= 2a+1$

$=$ Odd

$\therefore$ $2a+1$ is not divisible by '2'

$\Rightarrow$ x^2 is not divisible by '2'

$\Rightarrow$ x^2 is an odd integer

$p \rightarrow q$ is True

$\therefore$ the square of odd integer is odd

Ex :- 5

By using direct proofs method show that "every odd integer is the difference of two squares is odd"

Sol P : Assume 'n' is a odd integer and 'p' is True

q : Difference of two squares is odd $[P \rightarrow q \equiv T]$

$n = 2k+1$

$n = \underbrace{k^2 + (2k+1)}_{} - k^2$

$n = \underbrace{(k+1)^2}_{a} - \underbrace{k^2}_{b}$

$n = a^2 - b^2$

q is True

$\therefore P \rightarrow q \equiv T$

difference of Two Squares formula $(a^2 + b^2)$

we have derived 'q' is true

Every odd integer is the difference of Two squares.

Indirect Proof :-

3. Proof by Contrapositive :- $[P \rightarrow q \equiv \neg q \rightarrow \neg P]$

Automatically it is true — it becomes True

If we state that $P \Rightarrow q$ is logically equivalent to its Contrapositive $\neg q \Rightarrow \neg P$

Now To prove $P \Rightarrow q$ we assume that q is False and show that p is also False

Ex :- 1

Prove that if n^2 is odd then n is odd

Sol

p : n^2 is odd

q : n is odd

Mathematically $[P \longrightarrow q]$

The Contrapositive of $P \longrightarrow q$ is $\neg q \rightarrow \neg P$

$\neg q$: n is even, $\neg q$ is True

$\neg p$: n^2 is even

i.e $n = 2k$ where k is any positive integer

$n^2 = (2k)^2$

$n^2 = 4k^2$

$= 2(\underbrace{2k^2}_{a})$

$n^2 = 2a$

n^2 = even

$\therefore$ $\neg p$ is True

$\neg q \Rightarrow \neg p$

So, by logical equivalence $p \Rightarrow q$ is true

$\therefore$ n^2 is odd the n is odd

Ex :- 2

Prove that if (x, y belongs to Z) $x, y \in Z$ (set of Integers) such that xy is odd then both x and y are also odd

sol :- p : xy is odd

q : x and y are odd

$[p \rightarrow q]$

Now, by applying the method of Contra positive we can write the given statement as

$\neg q \rightarrow \neg p$

$\neg q$: x and y are even, $\neg q$ is True

$\neg p$: xy is even

Take $x = 2n$, $n \in Z$

$y = 2m$, $m \in Z$

$xy = 2n \cdot 2m$

$= 4nm$

$= 2(\underbrace{2nm}_{a})$

$\boxed{xy = 2a}$

xy is even

$\therefore p \to q \equiv T$

$\Rightarrow \neg p$ is True

$\neg q \to \neg p$ is True

By logically equvalence $p \Rightarrow q$ is True

if $x, y \in Z$, such that xy is odd the both x and y are also odd

Ex :- 3

Prove the (following statement).

" If $3n+2$ is odd 'n' is odd

sol :-

p : $3n+2$ is odd

q : n is odd

$p \to q$

The Contra positive of $p \to q$ is $[\neg q \to \neg p]$

we can write given statement as

$\neg q \to \neg p$

$\neg q$: n is even and it is True

$\neg p$: $3n+2$ is even

$n = 2k$

$3n+2 = 3(2k)+2$

$= 6k+2$

$= 2[\underbrace{3k+1}_{R}]$

$3n+2 = 2a$

$3n+2$ is even

$\therefore p \to q \equiv T$

$\neg p$ is True

By logically equivalence $p \to q$ is True

$\therefore$ if $3n+2$ is odd 'n' is odd

Ex :- 4

Show that if 'n' is an integer and n^3+5 is odd then 'n' is even integer by using Indirect proof

p : n is an integer, n^3+5 is odd

q : n is even integer

$p \to q$

By the method of Contrapositive we have to show

$\neg q \to \neg p$

$\neg q$: n is odd

$\neg p$: n is an integer, n^3+5 is Even

Assume $\neg q$ is True

$\Rightarrow$ n is odd is True

$\Rightarrow n = 2k+1$ for any integer 'k'

we have to show $\neg p$ is True

n^3+5 is even

$\Rightarrow (2k+1)^3+5$ $\boxed{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}$

$\Rightarrow [(2k)^3 + 3(2k)^2 \cdot 1 + 3 \cdot (2k) \cdot 1 + 1^3] + 5$

$\Rightarrow [8k^3 + 12k^2 + 6k + 1] + 5$

$\Rightarrow 8k^3 + 12k^2 + 6k + 6$

$\Rightarrow 2(\underbrace{4k^3+6k^2+3k+3}_{a})$

$\Rightarrow 2a$ even

So, n^3+5 is even which is True

$\therefore \neg q \Rightarrow \neg p$ is True

$\therefore$ if 'n' is an integer and n^3+5 is odd then 'n' is even

Ex :- 5

Show that If n is an integer and $5n+2$ is odd then 'n' is also odd integer by using indirect proof.

P : n is an integer, $5n+2$ is odd

q : n is odd integer

$P \rightarrow q$

By the method of Contrapositive we have to show $\neg q \Rightarrow \neg p$

$\neg q$: n is even integer

$\neg p$: n is an integer, $5n+2$ is even

Assume '$\neg q$' is True

$\Rightarrow$ n is even integer is True

$\Rightarrow$ ~~n=5k+2~~ $n = 2k$ for any integer k.

we have to show $\neg p$ is True

$5n+2$ is even

$\Rightarrow 5(2k)+2$

$\Rightarrow 10k+2$

$\Rightarrow 2(\underbrace{5k+1}_{a})$

$\Rightarrow 2a$

$\Rightarrow$ even

So, $5n+2$ is even which is True

$\therefore \neg q \Rightarrow \neg p$ is True

$\therefore$ If 'n' is an integer and $5n+2$ is odd then 'n' is also odd integer.

2. Proof by Contradiction [Indirect proof]

If $p \Rightarrow q$ is a statement, in this, we assume q is False [$\neg q$ is True]. Then by logical arguments we arrive to a situation where $\neg q$ is False. which is a Contradiction to the assumption so we conclude that 'q' must be True

Ex:- 1

Show that if $3n+2$ is odd then 'n' is odd by using Contradictory proof

p: $3n+2$ is odd

q: n is odd

we assume that q is False

i.e $\neg q$ is True

$\neg q$: n is even

$n = 2k$ for any integer k

$\Rightarrow 3n+2$

$= 3(2k)+2$

$= 6k+2$

$= 2(3k+1)$

$= 2a$

= even

So, $3n+2$ is even

This is a Contradictory statement for given 'p'

$\therefore$ we Conclude that $p \Rightarrow q$ is True

then 'n' is odd

if $3n+2$ is odd then n is odd

Ex :- 2

Show that for $x \in R$ if $x^3 + 4x = 0$ then $x = 0$ ↗ Rational number

sol :-

$p : x^3 + 4x = 0$

$q : x = 0$

By using Contradictory method, we assume $\neg q$ is True

i.e $x \neq 0$

$\neg q : x \neq 0$

$x^3 + 4x = 0$

$x(x^2+4) = 0$

$x^2 + 4 = 0$

$\boxed{x^2 = -4}$ is a Contradiction

[Any Negative number when it is square then it becomes positive]

Hence the given statement is True

i.e for $x \in R$ if $x^3 + 4x = 0$ then $x \neq 0$

Ex :- 3

By using proof by contradiction show that $\sqrt{2}$ is an Irrational number

sol :-

Assume $\sqrt{2}$ is a rational number a/b → they should not have common values

i.e $\left[\sqrt{2} = \dfrac{x}{y} \right]$ x, y has no Common factors

Squaring on both sides

$\Rightarrow (\sqrt{2})^2 = \left(\dfrac{x}{y}\right)^2$

$\Rightarrow 2 = \frac{x^2}{y^2}$

$\Rightarrow 2y^2 = x^2$

that means x^2 is even

which implies 'x' is even

$\Rightarrow x = 2k$, $k \in z$

$\Rightarrow x^2 = (2k)^2$

$= 4k^2$

$\therefore [x^2 = 2y^2]$

$\Rightarrow \cancel{2}y^2 = \cancel{4}2k^2$

$\Rightarrow y^2 = 2k^2$

so y^2 is even

which implies y is even

since x, y are both even integers that common factor is '2'

this is a contradiction to Assumed statement

$\therefore \sqrt{2}$ is an irrational number

2019 2020 SM Ex :- 4

prove that if $n = ab$ where a, b are positive integer then $a \leq \sqrt{n}$ (or) $b \leq \sqrt{n}$

sol

$p : n = ab$ where a, b are (two) positive integers

$q : a \leq \sqrt{n}$ (or) $b \leq \sqrt{n}$

we have to show that $\forall n$, $p(n) \longrightarrow q(n)$ is True.

Assume $q(n)$ is false

$\neg q : a \geq \sqrt{n}$ (or) $b \geq \sqrt{n}$

$\Rightarrow ab \geq \sqrt{n}\sqrt{n}$

$\Rightarrow ab > n$

$\Rightarrow \boxed{n \leq ab}$

But from the given Statement $n = ab$ to the obtain which is a contradiction result $(n < ab)$

$\therefore$ the given statement is True $\forall n$

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4. Induction proofs (or) proof by mathematical Induction

Mathematical induction is a technique of proving a statement, Theorem or Formula which is thought to be true for each and every Natural number values 'n'

By using generalizing this of the form of a principle which we use to prove any Mathematical statement is called as principle of Mathematical Induction.

The following are the steps (or) principles used to solve the Induction problems

Step 1 :- If the given statement is $P(n)$ where 'n' is a natural number then we show that the statement is True for $n = 1$

Step 2 :- Assume the given statement $P(n)$ is True for $n = k$ where k is any integer

Step 3 :- Now we prove the statement $P(n)$ is True for $n = k+1$ values

NOTE :-

"step 2" is called inductive Hypothesis

"step 3" is called inductive step

Ex :- 1

prove that the sum of cubes of 'n' natural numbers $= \left[\frac{n(n+1)}{2}\right]^2$

we have to show that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Step :- 1

Consider the above statement as $p(n)$

By using mathematical Induction we verify $p(n)$ is True or not by taking $n = 1$

$L \cdot H \cdot S = 1^3 = 1$ $\quad R \cdot H \cdot S = \left[\frac{1(1+1)}{2}\right]^2$

$= 1^2$

$= 1$

$\therefore L \cdot H \cdot S = R \cdot H \cdot S$

Result "is True for $p(n)$ when $n = 1$"

Step :- 2

Assume the given statement $p(n)$ is true for $n = k$ values where 'k' is any integer

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2 \longrightarrow (1)$$

Step :- 3

Now we have to prove that the statement $p(n)$ is also True for $n = k+1$ values

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{k+1(k+2)}{2}\right]^2$$

L.H.S :-

we know from ① that $\left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$ is called inductive step

$\Rightarrow \frac{k^2(k+1)^2}{4} + (k+1)^3$

$\Rightarrow \frac{k^2 \cdot (k+1)^2}{4} + \frac{4(k+1)^3}{4}$

$\Rightarrow \frac{(k+1)^2 (k^2+4k+4)}{4}$

$\Rightarrow \frac{(k+1)^2 (k+2)^2}{4}$

$\Rightarrow \left[\frac{(k+1)(k+2)}{2}\right]^2$

R.H.S

$\therefore$ P(n) is true for $n = k+1$

Hence by principle of mathematical Induction

" Sum of cubes of 'n' numbers $= \left(\frac{n(n+1)}{2}\right)$

2) Show that $1+3+5+ \dots + (2n-1) = n^2$

by using principle of mathematical Induction

proof :-

①

$P(n) = 1+3+5+ \dots + (2n-1) = n^2$

Step 1 :-

We have to check that P(n) is true for $n = 1$

LHS :- $P(1) = 1$ R.H.S :- $n^2 = (1)^2$

$\therefore$ LHS = R.H.S

so $p(n)$ is true for $n=1$

Step 2 :-

Assume $p(n)$ is true for $n=k$

$$P(k) = 1+3+5+\cdots+(2k-1) = k^2 \longrightarrow ①$$

Step 3 :-

we have to prove $p(n)$ is true for $n=k+1$

$$\Rightarrow 1+3+5+\cdots+(2(k+1)-1) = (k+1)^2$$

L.H.S

$$\Rightarrow 1+3+5+\cdots+(2k+2-1)$$

$$\underbrace{1+3+5+\cdots+2k-1}+2k+1$$

from eq ① this is equals to k^2

$$= k^2+2k+1$$

$$= (k+1)^2$$

$$= R.H.S$$

$L.H.S = R.H.S$

Hence by principle of mathematical induction

$p(n)$ is true for all values of n

$$1+3+5+\cdots+(2n-1) = n^2 \quad \forall n$$

3. S.T $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ by using principle of mathematical Induction

Given that

$$P(n) = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Step 1 :-

we have to check $p(n)$ is true for $n=1$

$$P(n) = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$P(1) = L.H.S = 1$$

$$R.H.S = \frac{1(1+1)}{2} = \frac{\cancel{2}}{\cancel{2}}$$

$$= 1$$

$\therefore$ R·H·S = R·H·S

$\therefore$ so p(n) is true for n = 1

Step 2 :-

Assume p(n) is true for n = K

$P(K) = 1+2+3+ \cdots + K = \frac{K(K+1)}{2} \longrightarrow (1)$

Step 3 :-

we have to prove p(n) is true for n = K+1

$\Rightarrow 1+2+3+ \cdots + K+1 = \frac{(K+1)(K+2)}{2}$

$\Rightarrow \underbrace{1+2+3+ \cdots + K} + K+1 = \frac{(K+1)(K+2)}{2}$

from eq (1) this is equal

L·H·S

$= \frac{K(K+1)}{2} + K+1$

$= \frac{(K^2+K)}{2} + K+1$

$= \frac{K^2+K+2(K+1)}{2}$

$= \frac{K^2+2K+K+2}{2}$

$= \frac{(K+1)(K+2)}{2}$

$= \frac{K(K+1)}{2} + K+1$

$= \frac{K(K+1)+2(K+1)}{2}$

$= \frac{(K+1)(K+2)}{2}$

R·H·S

L·H·S = R·H·S

Hence by principle of mathematical induction P(n) is True for all values of 'n'

$P(n) = 1+2+3+ \cdots + n = \frac{n(n+1)}{2}$

4. S.T $1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + 4\cdot 2^3 + \dots + n\cdot 2^{n-1} = (n-1)\cdot 2^n + 1$

Given that

$$p(n) = 1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + 4\cdot 2^3 + \dots + n\cdot 2^{n-1} = (n-1)2^n + 1 \longrightarrow (1)$$

Step 1 :-

we have to check p(n) is true for n=1

$p(1) = L\cdot H\cdot S = 1\cdot 2^{1-1} = 1\cdot 2^0 = 1$

$R\cdot H\cdot S = (1-1)2^0 + 1 = (0)\cdot 2^0 + 1$

$= 0 + 1$

$= 1$

$L\cdot H\cdot S = R\cdot H\cdot S$

So p(n) is true for n=1

Step 2 :-

Assume p(n) is true for n=K

$$p(K) = 1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + 4\cdot 2^3 + \dots + K\cdot 2^{K-1} = (K-1)2^K + 1 \longrightarrow (2)$$

Step 3 :-

we have to prove p(n) is true for n = K+1

$\Rightarrow p(K+1) = 1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + \dots + K+1\cdot 2^K = (K)2^{K+1} + 1$

$\Rightarrow 1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + \dots + K\cdot 2^{K-1} + K+1\cdot 2^K$

$\Rightarrow \underbrace{1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + \dots + K\cdot 2^{K-1}} + K+1\cdot 2^K$

from Equation (2) it is equalent to $(K-1)2^K + 1$

$\Rightarrow K\cdot 2^K - 2^K + 1 + (K+1)2^K$

$\Rightarrow K2^K - \cancel{2^K} + 1 + K\cdot 2^K + \cancel{2^K}$

$\Rightarrow K2^K + 1 + K\cdot 2^K$

$\Rightarrow 2K^{2K} + 1$

$\Rightarrow K2^{K+1} + 1$

$\Rightarrow R\cdot H\cdot S$

Result is True for p(n) when n=K+1

$\therefore$ By principle of mathematical Induction p(n) is True $\forall n$

5. Prove that by principle of mathematical Induction $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

$\forall$ positive Integers

Given that

$$p(n)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \rightarrow (1)$$

Step 1:-

we have to check $p(n)$ is true for $n=1$

$$p(1) = L\cdot H\cdot S = \frac{1}{1(1+1)}$$

$$= \frac{1}{2}$$

$$R\cdot H\cdot S = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

$$L\cdot H\cdot S = R\cdot H\cdot S$$

So $p(n)$ is true for $n=1$

Step 2 :-

Assume $p(n)$ is true for $n=k$

$$p(k) = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \rightarrow (2)$$

Step 3 :-

we have to prove $p(n)$ is true for $n=k+1$

$$\Rightarrow \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

L.H.S

$$\Rightarrow \underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}$$

from eq ② it is equalent to $\frac{k}{k+1}$

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$\frac{k(k+1)(k+2)}{(k+1)(k+2)} \Rightarrow \frac{k\cdot(k+2)+1}{(k+1)(k+2)}$$

$$\frac{k+1}{k+2} \Rightarrow \frac{k^2+2k+1}{(k+1)(k+2)} \Rightarrow \frac{(k+1)^2}{(k+1)(k+2)}$$

$$\Rightarrow \frac{k+1}{k+2} \quad R.H.S$$

R.H.S

Result is True for $p(n)$ when $n=k+1$

$\therefore$ By principle of mathematical Induction

$P(n)$ is True $\forall n$

6. prove that by principle of mathematical Induction

$$1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} \quad \forall n\geq 1$$

Given that

$$p(n) = 1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} \rightarrow ①$$

Step 1 :-

we have to check $p(n)$ is true for $n=1$

$$P(1) = L.H.S = (2(1)-1)^2$$
$$= 1$$

$$R.H.S = \frac{1(2(1)-1)(2(1)+1)}{3}$$

$$= \frac{\cancel{3}}{\cancel{3}}$$

$$= 1$$

L.H.S = R.H.S

So $P(n)$ is true for $n=1$

Step 2 :-

Assume $p(n)$ is true for $n=K$

$$P(K) = 1^2+3^2+5^2+\dots+(2K-1)^2 = \frac{K(2K-1)(2K+1)}{3} \rightarrow ②$$

$2k+2-1 \quad 2(k+1)+1$

$2k+2+1$

$2k+3$

Step 3 :-

we have to prove $p(n)$ is True for $n=k+1$

$$\Rightarrow 1^2+3^2+5^2+\dots+(2k+1)^2 = \frac{(K+1)(2k+1)(2k+3)}{3}$$

L.H.S

$$\Rightarrow \underbrace{1^2+3^2+5^2+\dots+(2k-1)^2}+(2k+1)^2$$

from eq ② it is equalent to $\frac{K(2K-1)(2k+1)}{3}$

$$\frac{K(2K-1)(2k+1)}{3}+(2k+1)^2$$

$$\frac{(2k^2-k)(2k+1)+3(2k+1)^2}{3}$$

$$\cancel{3k}\ \frac{4k^3+2k^2-2k^2-k+3(4k^2+1+4k)}{3}$$

$$\frac{4k^3+\cancel{2k^2}-\cancel{2k^2}-k+12k^2+3+12k}{3}$$

	4	12	11	3
1		-4	-8	-3
	4	8	3	0

$$\frac{4k^3+12k^2+11k+3}{3}$$

$$\frac{(k+1)(2k+1)(2k+3)}{3}$$

$= 4k^2+8k+3$

$4k^3+8k^2+3k$

$+4k^2+8k+3$

$4k^3+12k^2+11k+3$

Result is True for $P(n)$

when $n=k+1$

$\therefore$ By principle of mathematical induction
$P(n)$ is True.

7 Prove by mathematical induction $8^n - 3^n$ is a multipule of 5

Given that

$P(n) = 8^n - 3^n$ is a multipule of 5

Step 1:-

we have to check $P(n)$ is true for $n=1$

$P(1) = 8^1 - 3^1$

$= 8 - 3$

$= 5$

so $P(n)$ is true

Step 2:-

Assume

$P(k)$ is true for $n=k$

$P(k) = 8^k - 3^k$

$8^k - 3^k = 5m$

$8^k = 5m + 3^k \longrightarrow (2)$

Step 3:-

we have to prove $P(n)$ is true for $n = k+1$

$\Rightarrow 8^{k+1} - 3^{k+1}$ is a multiple of 5

$\Rightarrow 8^k \cdot 8^1 - 3^k \cdot 3$

$\Rightarrow [5m + 3^k] \cdot 8 - 3^k \cdot 3$

$\Rightarrow 8 \times 5m + 3^k \cdot 8 - 3 \cdot 3^k$

$\Rightarrow 8 \times 5m + (8-3)\, 3^k$

$\Rightarrow 8 \times 5m + 5 \cdot 3^k$

$\Rightarrow 5(8m + 3^k)$

is a multiple of 5

$P(k+1)$ is true

Hence by mathematical induction $8^n - 3^n$ is multiple of 5 $\forall n$

8. Show that n^3+2n is a multiple of 3 by using mathematical induction

Given that

$p(n) = n^3+2n$ is a multiple of 3

$= n^3+2n = 3 \rightarrow (1)$

Step 1 :-

we have to check $p(n)$ is true for $n=1$,

$p(1) = (1)^3 + 2(1)$

$= 1+2$

$= 3$

so $p(n)$ is true

Step 2 :-

Assume $p(n)$ is true for $n=k$

$p(k) = k^3+2k$

$k^3+2k = 3m$

$k^3 = 3m-2k \rightarrow (2)$

Step 3 :-

we have to prove $p(n)$ is true for $n=k+1$

$\Rightarrow (k+1)^3 + 2(k+1)$

$\Rightarrow k^3+3k^2+3k+1+2k+2$

$\Rightarrow (k^3+2k)+(3k^2+3k+3)$

$\Rightarrow (k^3+2k) + 3(k^2+k+1)$

multiple of 3

$\Rightarrow (3m - \cancel{2k} + \cancel{2k}) + 3(k^2+k+1)$

$\Rightarrow 3m + 3k^2+3k+3$

$\Rightarrow 3[m+k^2+k+1]$ is multiple of 3 fn

$p(k+1)$ is true

Hence by mathematical induction n^3+2n is multiple

9. Show that $6^{n+2} + 7^{2n+1}$ is Divisible by 43 by using mathematical induction

$P(n) = 6^{n+2} + 7^{2n+1}$ is a divisible by 43

$= 6^{n+2} + 7^{2n+1} \div 43$

Step 1 :-

we have to check $P(n)$ is true for $n = 1$

$P(1) = 6^{1+2} + 7^{2+1} \div 43$

$= 6^3 + 7^3 \div 43$

$= 216 + 343 \div 43$

$= 559 \div 43$

$\therefore P(1)$ is true

Step 2 :-

Assume $P(n)$ is true for $n = k$

$P(k) = 6^{k+2} + 7^{2k+1} = 43m$

$6^{k+2} = 43m - 7^{2k+1} \rightarrow (1)$

Step 3 :-

we have to prove $P(n)$ is true for $n = k+1$

$\Rightarrow 6^{(k+1)+2} + 7^{2(k+1)+1}$

$\Rightarrow 6^{k+3} + 7^{2k+3}$

$\Rightarrow 6^{k+2} \cdot 6 + 7^{2k+3}$ from (1) $\Rightarrow 6^{k+2} \cdot 6 + 7^{2k+1} \cdot 7^2$

$\Rightarrow [43m - 7^{2k+1}]\,6 + 7^{2k+1} \cdot 7^2$

$\Rightarrow 43m \cdot 6 + 7^{2k+1}[49 - 6]$

$\Rightarrow 43m \cdot 6 + 7^{2k+1} \cdot 43$

$\Rightarrow 43(6m + 7^{2k+1})$

$P(k+1)$ is true

Hence by principle of mathematical induction $P(n)$ is true all values of n

$\Rightarrow 6^{n+2} + 7^{2n+1}$ is a divisible by 43 $\forall n$

10 Show that $3^n + 7^n - 2$ is divisible by '8'

~~for~~ $\forall\, n \geq 1$

Given that

$p(n) = 3^n + 7^n - 2$ is divisible by '8'

Step 1 :- we have to check $p(n)$ is true for $n=1$

$P(1) = 3^1 + 7^1 - 2$

$= 10 - 2$

$= 8$

$\therefore$ $p(n)$ is true for $n=1$

Step 2 :-

Assume $p(n)$ is true for $n=k$

$P(k) = 3^k + 7^k - 2$

$3^k + 7^k - 2 = 8m$

$\cancel{3^k + 7^k = 8m + 2}$ $\quad 3^k = 8m - 7^k + 2 \rightarrow (2)$

Step 3 :-

we have to prove $p(n)$ is true for $n = k+1$

$\Rightarrow 3^{k+1} + 7^{k+1} - 2$

$\Rightarrow 3^k \cdot 3 + 7^k \cdot 7 - 2$

$\Rightarrow$ sub eq (2)

$\Rightarrow [8m - 7^k + 2] \cdot 3 + 7^k \cdot 7 - 2$

$\Rightarrow 24m - 3 \cdot 7^k + 6 + 7^k \cdot 7 - 2$

$\Rightarrow \cancel{24m - 3 \cdot 7^k + 6 + 7^k 5}$

$\Rightarrow 24m + 7^k [3 - 7] + 4$

$\Rightarrow 24m + (4) \cdot 7^k + 4$

$\Rightarrow \cancel{24m} \cdot 4 [6m + \underbrace{7^k + 1}_{24}]$ [Assume 7^k is odd then $7^k + 1$ becomes even $\Rightarrow 24$]

$+ 7^k \cdot 4$

$\Rightarrow 24m + 8y$

$\Rightarrow 8[3m+y]$ is divisible by 8

$\therefore P(K+1)$ is true for $n = K+1$

11) Prove by mathematical induction $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n} \;\forall\; n \geq 2$

Given that

$P(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n} \rightarrow (1)$

Step 1 :-

we have to check $P(n)$ is true for $n = 2$

$P(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$

$= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$

$= 1.727 > 1.414$

$L.H.S = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$

$= 1 + \frac{1}{\sqrt{2}}$

$= 1 + 0.72$

$= 1.72$

$= 2$

$R.H.S = \sqrt{n}$

$= \sqrt{2}$

$= 1.414$

$\therefore L.H.S > R.H.S$

$P(n)$ is true for $n = 2$

Step 2 :-

Assume $P(n)$ is true for $n = k$ Values

put $n = k$ in (1)

$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k} \longrightarrow (2)$

Step 3 :-

we have to prove $P(n)$ is true for $n = k+1$

$$P(k+1) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} \text{ is true}$$

$$\Rightarrow \underbrace{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}}}_{> \sqrt{k} \text{ from } 1} + \frac{1}{\sqrt{k+1}}$$

$$\Rightarrow > \sqrt{k} + \frac{1}{\sqrt{k+1}} \leftarrow 1$$

$$\Rightarrow > \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}}$$

$$\Rightarrow > \frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}}$$

$$\Rightarrow > \frac{\sqrt{k \cdot k} + 1}{\sqrt{k+1}} \quad [\because (k+1) > k]$$

$$\Rightarrow > \frac{k+1}{\sqrt{k+1}}$$

$$\Rightarrow > \frac{\cancel{(\sqrt{k+1})}(\sqrt{k+1})}{\cancel{\sqrt{k+1}}}$$

$\Rightarrow$ $P(k+1)$ is true

Hence by principle of mathematical induction the given statement $P(n)$ is true $\forall n \geq 2$

dividend = (division)

divided will ... of division so we take

12. Show that $11^{n+2} + 12^{2n+1}$ is divisible by 133 using mathematical induction. $\forall n \geq 0$

Given that

$P(n) = 11^{n+2} + 12^{2n+1}$ is divisible by 133

Step 1 :-

we have to check $P(n)$ is true for $n=1$

$P(1) = 11^{1+2} + 12^{2+1}$

$= 11^3 + 12^3$

$= 1331 + 1728$

$= 3059$

$P(0) = 11^2 + 11$

$= 121 + 11$

$= 133$

$133)\,3059\,(23$

3059 is divisible by 133

$P(n)$ is true for $n=1$

Step 2 :-

Assume $P(n)$ is true for $n=k$

$P(k) = 11^{k+2} + 12^{2k+1}$

$11^{k+2} + 12^{2k+1} = 133m$

$11^{k+2} = 133m - 12^{2k+1} \rightarrow (1)$

Step 3 :-

we have to prove $P(n)$ is true for $n = k+1$

$\Rightarrow 11^{k+1+2} + 12^{2(k+1)+1}$

$\Rightarrow 11^{k+3} + 12^{2k+3}$

$\Rightarrow 11^{k+2} \cdot 11 + 12^{2k+1} \cdot 12^2$ from (1)

$\Rightarrow (133m - 12^{2k+1})\,11 + 12^{2k+1}\,12^2$

$\Rightarrow (133m)\cdot 11 - (12^{2k+1})\cdot 11 + 12^{2k+1}\, 144$

$\Rightarrow 133m \cdot 11 + 12^{2k+1}[144 - 11] \Rightarrow 133m \cdot 11 + 12^{2k+1} \cdot 133$

$\Rightarrow 133[11m + 12^{2k+1}]$

$P(k+1)$ is true

Hence by principle of mathematical induction $P(n)$ is true $\forall n \geq 0$

13 Show that n^5-n is divisible by '5' $\forall n \geq 1$ by using mathematical induction

Given that

$P(n) = n^5 - n$ is divisible by 5

step 1 :-

we have to check $P(n)$ is true for $n=2$

$P(2) = 2^5 - 2$

$= 32 - 2$

$= 30$

$\left\{ P(1) = 1^5 - 1 = 0 \right\}$

It is divisible by '5'

$P(n)$ is true for $n=2$

step 2 :-

Assume $P(n)$ is true for $n=K$

$P(K) = K^5 - K$

$K^5 - K = 5m$

$K^5 = 5m + K \rightarrow (1)$

step 3 :-

we have to prove $P(n)$ is true for $n = K+1$

$\Rightarrow (K+1)^5 - (K+1)$

$\Rightarrow \left[K^5 + 5C_1 K^{5-1} \cdot 1 + 5C_2 K^{5-2} 1^2 + \cdots + 5C_5 K^{5-5} 1^5\right] - (K+1)$

Formula :-

$(x+a)^n = x^n + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \cdots + nC_n x^{n-n} a^n$

$\Rightarrow K^5 + 5K^4 + 10K^3 + 10K^2 + 5K + \cancel{1} - K - \cancel{1}$

$\boxed{nC_r = \dfrac{n!}{r!(n-r)!}}$

from 1

$(5m+k)+5k^4+10k^3+10k^2+5k-k$

$5[m+k^4+2k^3+2k^2+k]$

is divisible of 5

R.H.S

P(k+1) is true

Hence by principle of mathematical induction

n^5-n is divisible by 5 $\forall n \geq 1$

12. Show that mathematical induction of $a+ar+ar^2+\cdots+ar^n$

$= \dfrac{ar^{n+1}-a}{r-1}$ where $r \neq 1$, $n \geq 0$

Geometric Progression

Given that

$P(n) = a+ar+ar^2+ar^3+\cdots+ar^n = \dfrac{ar^{n+1}-a}{r-1}$

Step 1 :-

we have to check p(n) is true for $n=0$

L.H.S

$P(0) = ar^n$

$= ar^0$

$= a$

R.H.S

$P(0) = \dfrac{ar^{0+1}-a}{r-1}$

$= \dfrac{ar-a}{r-1}$

$= \dfrac{a(\cancel{r-1})}{\cancel{r-1}}$

$= a$

L.H.S = R.H.S

P(n) is true for $n=0$

Step 2 :-

Assume p(n) is true for $n=k$

$P(k) = a+ar+ar^2+ar^3+\cdots+ar^k = \dfrac{ar^{k+1}-a}{r-1} \to (1)$

Step 3 :-

we have to prove $p(n)$ is true for $n = k+1$

$$\Rightarrow a + ar + ar^2 + ar^3 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r-1}$$

L.H.S :-

$$\Rightarrow \underbrace{a + ar + ar^2 + ar^3 + \cdots + ar^k}_{\text{from (1)}} + ar^{k+1}$$

$$\Rightarrow \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$\Rightarrow \frac{ar^{k+1} - a + (r-1)\, ar^{k+1}}{r-1}$$

$$\Rightarrow \frac{\cancel{ar^{k+1}} - a + ar^{k+2} - \cancel{ar^{k+1}}}{r-1}$$

$$\Rightarrow \frac{-a + ar^{k+2}}{r-1}$$

$$\Rightarrow \frac{ar^{k+2} - a}{r-1} = R.H.S$$

L.H.S = R.H.S

$\therefore$ $p(n)$ is True for $n = k+1$

By the principle of mathematical induction

$P(n)$ is true $\forall\, n \geq 0$

15 Show that $n < 2^n$ $\forall$ positive integers of 'n' $\forall n \geq 1$

Given that

$p(n) = n < 2^n \; \forall n \geq 1 \longrightarrow (1)$

Step 1 :-

we have to check p(n) is true for n=1

$p(1) = 1 < 2^1$

$1 < 2$

p(n) is true for n=1

Step 2

Assume p(n) is true for n=k values

put n=k in (1)

$\Rightarrow P(K) = K < 2^K \longrightarrow (2)$

Step 3 :-

we have to show that p(n) is true for n=K+1

$P(K+1) \Rightarrow (K+1) < 2^{K+1}$ is true

L.H.S :-

$(K+1) \Rightarrow < 2^{K+1}$

$\Rightarrow < 2^K + 1$

[from (2) $K < 2^K$] $\Rightarrow < 2^K + 2^K$ $[\because 1 < 2^K$ where $K > 0]$

$\Rightarrow < 2 \cdot 2^K$ $[a^{x+y} = a^x \cdot a^y]$

$\Rightarrow < 2^{K+1}$

R.H.S

$\therefore (K+1) < 2^{K+1}$

Hence L.H.S = R.H.S

$\therefore$ By principle of mathematical induction the given statement p(n) is true $n \geq 1$

16 Show that $2^n < n!$ if $n \geq 4$

Given that

$P(n) = 2^n < n! \ \forall n \geq 4$

Step 1:-

we have to check $P(n)$ is true for $n=4$

$P(4) = 2^4 < 4!$ $\qquad$ $P(5) = 2^5 < 5!$

$= 16 < 4 \times 3 \times 2 \times 1$ $\qquad$ $= 32 < 5 \times 4 \times 3 \times 2 \times 1$

$= 16 < 24$ $\qquad$ $= 32 < 120$

$P(n)$ is true for $n=4$

Step 2:-

Assume $P(n)$ is true for $n=k$

$P(k) = 2^k < k! \rightarrow (1)$

Step 3:-

we have to prove $P(n)$ is true for $n=k+1$

$P(k+1) = 2^{k+1} < (k+1)!$ is true

L.H.S :- $\qquad$ L.H.S $= 2^{k+1}$

$2^{k+1} \leq 2^k \cdot 2$ $\qquad$ $\leq 2^k \cdot 2$

$< 2 \cdot k!$ $\qquad$ $< 2 \cdot k!$

$\qquad$ [$\because 2 < k+1$, where $k \geq 4$]

$< (k+1) k!$ $\qquad$ $< (k+1) \cdot k!$

$2^{k+1} < (k+1)!$ $\qquad$ $< (k+1)!$

R.H.S

By principle of mathematical induction

the given statement $P(n)$ is true $\forall n \geq 4$

17) show that $4n < n^2 - 7 \ \forall \ n \geq 6$

$P(n) = 4n < n^2 - 7$

step1 :-

we have to check $p(n)$ is true for $n = 6$

$P(6) = 4(6) < 6^2 - 7$

$= 24 < 36 - 7$

$= 24 < 29$

True

$\therefore P(6)$ is True for $p(n)$

Step 2 :-

Assume $p(K)$ is true for $n = K$

$P(K) = 4K < K^2 - 7 \rightarrow (1)$

step3 :-

we have to prove $p(n)$ is true for $n = K+1$

$P(K+1) = 4(K+1) < (K+1)^2 - 7$

$L.H.S = 4(K+1)$

$\leq 4K + 4$ $\{ 4 < 2K+1 \ \forall \ K \geq 6 \}$

$< (K^2 - 7) + 4$ 2K cuz, k is even

$< (K^2 - 7) + 2K + 1$

$< K^2 - 7 + 2K + 1$

$< K^2 + 2K + 1 - 7$

$< (K+1)^2 - 7$

R.H.S

Strong Induction proofs :-

If the given preposition depends on more than one basic Condition or initial Conditions then we solve these problems by using Strong Induction Method (Second principle of Mathematical induction)

Examples :-

2020

1. Prove that $u_n = 3^n - 2^n \ \forall n \geq 1$ where $u_1 = 1$ $u_2 = 5$ and $u_{n+1} = 5u_n - 6u_{n-1} \ \forall n \geq 2$

Proof :-

Given that

$P(n) = 3^n - 2^n \ \forall n \geq 1 \rightarrow (1)$

$u_1 = 1$ $u_2 = 5$ and $u_{n+1} = 5u_n - 6u_{n-1}$

Step :- 1

we have to verify p(n) is true for n=1 and n=2

$P(1) = 3^1 - 2^1$
$= 3 - 2$
$= 1$
u_1 True

$P(2) = 3^2 - 2^2$
$= 9 - 4$
$= 5$
u_2 True

$\therefore$ p(n) is true for n = 1 & 2

Step :- 2

Assume that p(n) is true for n = k

$u_k = 3^k - 2^k \ \forall n \geq 1 \rightarrow (2)$

is true

Step 3 :-

Now we have to prove p(n) is true for n = k+1

$p(k+1) = U_{k+1} = 3^{k+1} - 2^{k+1}$

we know that $u_{n+1} = 5u_n - 6u_{n-1}$

$\Rightarrow u_{k+1} = 5u_k - 6u_{k-1}$

Sub eq (2)

$= 5[3^k - 2^k] - 6u_{k-1}$

Sub u_{k-1} in eq (1) we get $u_{k-1} = 3^{k-1} - 2^{k-1}$

$= 5[3^k - 2^k] - 6[3^{k-1} - 2^{k-1}]$

$= 5(3^k) - 5(2^k) - 6(3^k)\frac{1}{3} + 6(2^k)\frac{1}{2}$

$= 5(3^k) - 5(2^k) - 2\cdot 3^k + 3\cdot 2^k$

$= 3^k[5-2] - 2^k(5-3)$

$= 3\cdot 3^k - 2\cdot 2^k$

$= 3^{k+1} - 2^{k+1}$

R.H.S

$\Rightarrow$ P(k+1) is true

By the second method of principle of mathematical induction the given statement P(n) is true $\forall n \geq 1$

2) using Mathematical induction prove that $(a-b)$ is a factor of $a^n - b^n$ for all positive integer n

Given that $P(n)$: $a-b$ is a factor of $a^n - b^n$

Step 1 :-

we have to prove P(n) is true for 'n' greater than (or) equals to 1

$P(1) = a^1 - b^1$

$= a - b$ is a factor of $(a-b)$ is True

$P(2) = a^2 - b^2$

$= (a+b)(a-b)$

$= (a-b)$ is a factor of $P(n) = 2$

Step 2 :-

Assume $p(n)$ is true for $n=k$ values

$\Rightarrow P(k) = a^k - b^k$ is true $\forall k$

Step 3 :-

Now we have to show $p(k+1)$ is true

$\Rightarrow (a-b)$ is a factor of $a^{k+1} - b^{k+1}$ is true

$a^{k+1} - b^{k+1} = (a+b)(a^k - b^k) + ab^k - ba^k$

$\Rightarrow a+b \underbrace{(a^k - b^k)}_{(a-b)x} - ab \underbrace{(a^{k-1} - b^{k-1})}_{(a-b) \cdot y}$

$p(k) = a^k - b^k = (a-b)x$

$\Rightarrow a+b [(a-b)x - ab(a-b)y]$

$\Rightarrow (a-b)[(a+b)x - (ab)y]$

$p(k-1) = a^{k-1} - b^{k-1} = (a-b)y$

is a factor of $(a-b)$

$\therefore p(k+1)$ is true for $n = k+1$

Hence by 2^{nd} method of mathematical induction $(a-b)$ is a factor of $a^n - b^n \ \forall\ n \geq 1$

3) By using 2^{nd} principle of Mathematical Induction show that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is divisible by 2^n $\forall\ n \geq 0$

Given that

$p(n) = (3+\sqrt{5})^n + (3-\sqrt{5})^n$ is divisible by 2^n

Step 1 :-

we have to show that $p(n)$ is true when $n = 0, 1$

$p(0) = (3+\sqrt{5})^0 + (3-\sqrt{5})^0$ R.H.S $= 2^0$

$= 1+1$ $\quad = 2^0$

$= 2$ $\quad = 1$

2 is divisible by 1

so, $P(0)$ is true

L.H.S $\quad$ R.H.S

$P(1) = (3+\sqrt{5})^1 + (3-\sqrt{5})^1$ $\quad P(1) = 2^1$

$= 3+\sqrt{5} + 3 - \sqrt{5}$ $\quad = 2$

$= 6$

6 is divisible by 2

so, $P(1)$ is true

Step 2 :-

Assume $P(n) = k$ is true

$P(k) = (3+\sqrt{5})^k + (3-\sqrt{5})^k$ is divisible by 2^k is true

Step 3 :-

Now we have to prove $p(n)$ is ~~prove~~ true for $n = k+1$

i.e

$P(k+1) = (3+\sqrt{5})^{k+1} + (3-\sqrt{5})^{k+1}$ is divisible by 2^{k+1} is true

Assume $a = 3+\sqrt{5}$, $b = 3-\sqrt{5}$

then $a+b = 3+\sqrt{5}+3-\sqrt{5}$

$= 6$

$ab = (3+\sqrt{5})(3-\sqrt{5})$ $[(a+b)(a-b) = a^2-b^2]$

$= 3^2-(\sqrt{5})^2$

$= 9-5$

$= 4$

L.H.S $= (3+\sqrt{5})^{k+1}+(3-\sqrt{5})^{k+1}$

$= (a)^{k+1}+(b)^{k+1}$

$= (a+b)(a^k+b^k)+ab^k-ba^k$

$ab+a-b$

$= (a+b)\underbrace{(a^k+b^k)}_{2^k \cdot x}-ab\underbrace{(a^{k-1}+b^{k-1})}_{2^{k-1}y}$

[$\because$ $P(k)$ is divisible by $2^k \Rightarrow a^k+b^k = 2^k \cdot x$]

[$\because$ $P(k-1)$ is divisble by $2^{k-1} \Rightarrow a^{k-1}+b^{k-1} = 2^{k-1}y$]

$= (a+b)\,2^k \cdot x - ab(2^{k-1}y)$

$= 6 \cdot 2^k \cdot x - 4(2^{k-1}y)$

$= 3 \cdot 2 \cdot 2^k \cdot x - 2 \cdot 2 \cdot 2^{k-1}y$

$= 3 \cdot 2^{k+1}x - 2^{k+1}y$

$= 2^{k+1}[3x-y]$

is divisible by 2^{k+1}

$P(k+1)$ is true

4 By using 2nd principle of Mathematical induction Show that nth fibonacci series is $F_n \ge \left[\frac{1+\sqrt{5}}{2}\right]^{n-2}$ $\forall\, n \ge 3$

Assume $F_0 = 0$, $F_1 = 1$ and $F_2 = 1$

Given than $p(n) : F_n \ge \left[\frac{1+\sqrt{5}}{2}\right]^{n-2}$ $\forall\, n \ge 3$

Step :-1

we have to show that $p(n)$ is true for $n = 3, 4$

$$p(3) = F_3 = F_2 + F_1 \ge \left(\frac{1+\sqrt{5}}{2}\right)^{3-2}$$
$$= 1+1 \ge \left(\frac{1+\sqrt{5}}{2}\right)^{1}$$
$$= 2 \ge \left(\frac{3.236}{2}\right) = 1.61 \text{ (approx)}$$

$P(3)$ is true

Similarly we have to show $p(4)$ is True

$$p(4) = F_4 = F_3 + F_2$$
$$= 2+1$$
$$= 3 \ge \left(\frac{1+\sqrt{5}}{2}\right)^{4-2}$$
$$= \left(\frac{1+\sqrt{5}}{2}\right)^{2}$$
$$= 3 \ge (1.61)(1.61)$$
$$= 3 \ge 2.97$$
$$= 3 \ge 3 \text{ is true}$$

$p(4)$ is true

Step 2 :-

Assume $p(n)$ is true for $n = k$ values

~~Fk~~ $F_k \ge \left[\frac{1+\sqrt{5}}{2}\right]^{k-2}$

Let $\alpha = \left[\frac{1+\sqrt{5}}{2}\right]$

$\Rightarrow F_k \geq \alpha^{k-2} \rightarrow$ (1)

Step 3 :-

we have to show $p(n)$ is true for $n=k+1$

$\Rightarrow p(k+1) : F_{k+1} \geq \left[\frac{1+\sqrt{5}}{2}\right]^{k-1}$

Let

$$\alpha = \frac{1+\sqrt{5}}{2}$$

$$2\alpha = 1+\sqrt{5}$$

$$2\alpha - 1 = \sqrt{5}$$

$$(2\alpha-1)^2 = (\sqrt{5})^2$$

$$4\alpha^2 - 4\alpha + 1 = 5$$

$$4\alpha^2 - 4\alpha + 1 - 5 = 0$$

$$4(\alpha^2 - \alpha - 1) = 0$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha^2 = \alpha + 1$$

Now R.H.S :-

$$\left[\frac{1+\sqrt{5}}{2}\right]^{k-1}$$

$$= \alpha^{k-1}$$

$$= \alpha^2 \cdot \alpha^{k-3}$$

$$= (\alpha+1)\alpha^{k-3}$$

$$= \alpha^{k-2} + \alpha^{k-3}$$

from (1) w.r.t,

$F_k \geq \alpha^{k-2}$ and $F_{k-1} > \alpha^{k-3}$ are true

$$\therefore F_{k+1} \geq \alpha^{k-2} + \alpha^{k-3}$$

$$F_{k+1} \geq \alpha^{k-1}$$

$$F_{k+1} \geq \left[\frac{1+\sqrt{5}}{2}\right]^{k-1}$$

Hence by 2^{nd} principle of Mathematical Induction
$p(n)$ is true $\forall n \geq 3$

Well Ordering principle :-

Every non-empty set of integers Contian atleast element

Every non-empty Subsett of N has a smallest element

Ex :- $\{-3, -2, -1\}$ ⇒ The well ordered
⇒ { Smallest element is -3 }

$\{5, 4, 3, 2, 1, 0, -1 ---\}$ ⇒ Not well ordered
{Since it is in decreasing and we Cannot identify what is the least element in the set }

TO prove well ordering using Induction :-
(i) show that P_1 is True
(ii) Assume that P_k is true
(iii) show that $P(k+1)$ is true if P_k is true

Ex :- prove that $2^{n+2} + 3^{2n+1}$ is divisible by 7

proof :- (i) :- show that P_1 is true
∵ 'n' is an element of Positive integer, the Smallest element $= 1$

⇒ $2^{n+2} + 3^{2n+1}$

⇒ $2^{1+2} + 3^{2(1)+1}$

⇒ $2^3 + 3^3$

⇒ $8 + 27$

⇒ 35

which is divisible by 7

∴ $p(1)$ is true for $n = 1$

or - sum
And - product

(ii) Assume P_k is True for $n=k$

$P(k) = 2^{k+2} + 3^{2k+1}$ is divisible by 7

$2^{k+2} + 3^{2k+1} = 7m$

$2^{k+2} = 7m - 3^{2k+1} \rightarrow (1)$

$\therefore$ $p(n)$ is true for $n=k$

(iii) we have to prove that $p(n)$ is true for $n = k+1$

$P(k+1) = 2^{k+1+2} + 3^{2(k+1)+1}$ is divisible by 7

$= 2^{k+3} + 3^{2k+3}$

$= 2^{k+2} \cdot 2 + 3^{2k+3}$

from eq (1)

$(7m - 3^{2k+1}) 2 + 3^{2k+3}$

$7 \cdot 2m - 2\, 3^{2k+1} + 3^{2k+1}\, 3^{2}$

$7 \cdot 2m + 3^{2k+1} (9-2)$

$7 \cdot 2m + 3^{2k+1}\, 7$

$7(2m + 3^{2k+1})$ is divisible by 7

$\therefore$ $p(n)$ is True for $n = k+1$

Hence $2^{n+2} + 3^{2n+1}$ is divisible by 7.

Recursion :-

Solve the main problem by using solution of simple sub problems of same type

Eg:- $N! = N(N-1)!$

$5! = 5(5-1)!$

$\Rightarrow$ fact (5) = 5 * fact (4) $5 \times 24 = 120$

$\Rightarrow$ fact (4) = 4 * fact (3) $4 \times 3 \times 2 = 24$

$\Rightarrow$ fact (3) = 3 * fact (2) $3 \times 2 = 6$

$\Rightarrow$ fact (2) = 2 * fact (1) $= 2$

$\Rightarrow$ fact (1) = 1 * fact (0)

$\Rightarrow$ fact (0) = 1

An algorithm is called recursive if it solve a problem by reducing it to instance of same problem with small inputs

Eg:- find gcd (8, 13)

if a = 0 then
return b
else
return gcd (1 mod a, a)
end if
{ output is gcd (a, b) }
	gcd (8, 13)
	$\Rightarrow$ gcd (13, mod 8, 8) $\Rightarrow$ (5, 8)
	$\Rightarrow$ gcd (8 mod 5, 5) $\Rightarrow$ (3, 5)
	$\Rightarrow$ gcd (5 mod 3, 3) $\Rightarrow$ (2, 3)
	$\Rightarrow$ gcd (3 mod 2, 2) $\Rightarrow$ (1, 2)
	$\Rightarrow$ gcd (0 mod 1, 1) $\Rightarrow$ (0, 1)

here a = 0 then 1 is returned

$\therefore$ gcd (8, 13) = 1