



Unit-1

Logics and Proofs

Logic

- Crucial for mathematical reasoning
- Important for program design
- Used for designing electronic circuitry
- (Propositional)Logic is a system based on propositions.
- A proposition is a (declarative) statement that is either true or false (not both).
- We say that the truth value of a proposition is either true (T) or false (F).
- Corresponds to 1 and 0 in digital circuits

The Statement/Proposition Game

"Elephants are bigger than mice."

Is this a statement? yes

Is this a proposition? yes

What is the truth value
of the proposition? true

The Statement/Proposition Game

"520 < 111"

Is this a statement? yes

Is this a proposition? yes

What is the truth value
of the proposition? false

The Statement/Proposition Game

" $y > 5$ "

Is this a statement? yes

Is this a proposition? no

Its truth value depends on the value of y ,
but this value is not specified.

We call this type of statement a
propositional function or open sentence.

The Statement/Proposition Game

"Today is January 27 and $99 < 5$."

Is this a statement? yes

Is this a proposition? yes

What is the truth value
of the proposition? false

The Statement/Proposition Game

"Please do not fall asleep."

Is this a statement? no

It's a request.

Is this a proposition? no

Only statements can be propositions.

The Statement/Proposition Game

"If the moon is made of cheese,
then I will be rich."

Is this a statement? yes

Is this a proposition? yes

What is the truth value
of the proposition? probably true

The Statement/Proposition Game

" $x < y$ if and only if $y > x$."

Is this a statement? yes

Is this a proposition? yes

... because its truth value
does not depend on
specific values of x and y .

What is the truth value
of the proposition? true

Combining Propositions

As we have seen in the previous examples, one or more propositions can be combined to form a single compound proposition.

We formalize this by denoting propositions with letters such as p, q, r, s , and introducing several logical operators or logical connectives.

Logical Operators (Connectives)

We will examine the following logical operators:

- Negation (NOT, \neg)
- Conjunction (AND, \wedge)
- Disjunction (OR, \vee)
- Exclusive-or (XOR, \oplus)
- Implication (if - then, \rightarrow)
- Biconditional (if and only if, \leftrightarrow)

Truth tables can be used to show how these operators can combine propositions to compound propositions.

Negation (NOT)

Unary Operator, Symbol: \neg

P	$\neg P$
true (T)	false (F)
false (F)	true (T)

Conjunction (AND)

Binary Operator, Symbol: \wedge

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)

Binary Operator, Symbol: \vee

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or (XOR)

Binary Operator, Symbol: \oplus

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication (if - then)

Binary Operator, Symbol: \rightarrow

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional (if and only if)

Binary Operator, Symbol: \leftrightarrow

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Statements and Operators

Statements and operators can be combined in any way to form new statements.

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Statements and Operations

Statements and operators can be combined in any way to form new statements.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Exercises

- To take discrete mathematics, you must have taken calculus or a course in computer science.
- When you buy a new car from Acme Motor Company, you get \$2000 back in cash or a 2% car loan.
- School is closed if more than 2 feet of snow falls or if the wind chill is below -100.

Exercises

- To take discrete mathematics, you must have taken calculus or a course in computer science.
 - P: take discrete mathematics
 - Q: take calculus
 - R: take a course in computer science
- $P \rightarrow Q \vee R$
- Problem with proposition R
 - What if I want to represent "take CMSC201"?

Exercises

- When you buy a new car from Acme Motor Company, you get \$2000 back in cash or a 2% car loan.
 - P: buy a car from Acme Motor Company
 - Q: get \$2000 cash back
 - R: get a 2% car loan
- $P \rightarrow Q \oplus R$
- Why use XOR here? - example of ambiguity of natural languages

Exercises

- School is closed if more than 2 feet of snow falls or if the wind chill is below -100.
 - P: School is closed
 - Q: 2 feet of snow falls
 - R: wind chill is below -100
- $Q \wedge R \rightarrow P$
- Precedence among operators:
 $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

Equivalent Statements

P	Q	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

The statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are **logically equivalent**, since they have the same truth table, or put it in another way, $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

Tautologies and Contradictions

A tautology is a statement that is always true.

Examples:

- $R \vee (\neg R)$
- $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

A contradiction is a statement that is always false.

Examples:

- $R \wedge (\neg R)$
- $\neg(\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q))$

The negation of any tautology is a contradiction, and the negation of any contradiction is a tautology.

Equivalence

Definition: two propositional statements $S1$ and $S2$ are said to be (logically) equivalent, denoted $S1 \equiv S2$ if

- They have the same truth table, or
- $S1 \Leftrightarrow S2$ is a tautology

Equivalence can be established by

- Constructing truth tables
- Using equivalence laws (Table 5 in Section 1.2)

Equivalence

Equivalence laws

- Identity laws, $P \wedge T \equiv P,$
- Domination laws, $P \wedge F \equiv F,$
- Idempotent laws, $P \wedge P \equiv P,$
- Double negation law, $\neg(\neg P) \equiv P$
- Commutative laws, $P \wedge Q \equiv Q \wedge P,$
- Associative laws, $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R,$
- Distributive laws, $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R),$
- De Morgan's laws, $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
- Law with implication $P \rightarrow Q \equiv \neg P \vee Q$

Exercises

- Show that $P \rightarrow Q \equiv \neg P \vee Q$: by truth table
- Show that $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$:
by equivalence laws (q20, p27):
 - Law with implication on both sides
 - Distribution law on LHS

Propositional Functions & Predicates

Propositional function (open sentence):
statement involving one or more variables,

e.g.: $x - 3 > 5$.

Let us call this propositional function $P(x)$, where
 P is the predicate and x is the variable.

What is the truth value of $P(2)$? false

What is the truth value of $P(8)$? false

What is the truth value of $P(9)$? true

When a variable is given a value, it is said to be
instantiated

Truth value depends on value of variable

Propositional Functions

Let us consider the propositional function $Q(x, y, z)$ defined as:

$$x + y = z.$$

Here, Q is the predicate and x , y , and z are the variables.

What is the truth value of $Q(2, 3, 5)$? true

What is the truth value of $Q(0, 1, 2)$? false

What is the truth value of $Q(9, -9, 0)$? true

A propositional function (predicate) becomes a proposition when all its variables are instantiated.

Propositional Functions

Other examples of propositional functions

$\text{Person}(x)$, which is true if x is a person

$\text{Person}(\text{Socrates}) = \text{T}$

$\text{Person}(\text{dolly-the-sheep}) = \text{F}$

$\text{CSCourse}(x)$, which is true if x is a computer science course

$\text{CSCourse}(\text{CMSC201}) = \text{T}$

$\text{CSCourse}(\text{MATH155}) = \text{F}$

How do we say

All humans are mortal

One CS course

Universal Quantification

Let $P(x)$ be a predicate (propositional function).

Universally quantified sentence:

For all x in the universe of discourse $P(x)$ is true.

Using the universal quantifier \forall :

$\forall x P(x)$ "for all $x P(x)$ " or "for every $x P(x)$ "

(Note: $\forall x P(x)$ is either true or false, so it is a proposition, not a propositional function.)

Universal Quantification

Example: Let the universe of discourse be all people

$S(x)$: x is a UMBC student.

$G(x)$: x is a genius.

What does $\forall x (S(x) \rightarrow G(x))$ mean?

"If x is a UMBC student, then x is a genius." or

"All UMBC students are geniuses."

If the universe of discourse is all UMBC students, then the same statement can be written as

$\forall x G(x)$

Existential Quantification

Existentially quantified sentence:

There exists an x in the universe of discourse for which $P(x)$ is true.

Using the existential quantifier \exists :

$\exists x P(x)$ "There is an x such that $P(x)$."

"There is at least one x such that $P(x)$."

(Note: $\exists x P(x)$ is either true or false, so it is a proposition, but no propositional function.)

Existential Quantification

Example:

$P(x)$: x is a UMBC professor.

$G(x)$: x is a genius.

What does $\exists x (P(x) \wedge G(x))$ mean ?

"There is an x such that x is a UMBC professor and x is a genius."

or

"At least one UMBC professor is a genius."

Quantification

Another example:

Let the universe of discourse be the real numbers.

What does $\forall x \exists y (x + y = 320)$ mean ?

"For every x there exists a y so that $x + y = 320$."

Is it true?

yes

Is it true for the natural numbers?

no

Disproof by Counterexample

A counterexample to $\forall x P(x)$ is an object c so that $P(c)$ is false.

Statements such as $\forall x (P(x) \rightarrow Q(x))$ can be disproved by simply providing a counterexample.

Statement: "All birds can fly."

Disproved by counterexample: Penguin.

Negation

$\neg(\forall x P(x))$ is logically equivalent to $\exists x (\neg P(x))$.

$\neg(\exists x P(x))$ is logically equivalent to $\forall x (\neg P(x))$.

See Table 2 in Section 1.3.

This is de Morgan's law for quantifiers

Negation

Examples

Not all roses are red

$$\neg \forall x (\text{Rose}(x) \rightarrow \text{Red}(x))$$

$$\exists x (\text{Rose}(x) \wedge \neg \text{Red}(x))$$

Nobody is perfect

$$\neg \exists x (\text{Person}(x) \wedge \text{Perfect}(x))$$

$$\forall x (\text{Person}(x) \rightarrow \neg \text{Perfect}(x))$$

Nested Quantifier

A predicate can have more than one variables.

- $S(x, y, z)$: z is the sum of x and y
- $F(x, y)$: x and y are friends

We can quantify individual variables in different ways

- $\forall x, y, z (S(x, y, z) \rightarrow (x \leq z \wedge y \leq z))$
- $\exists x \forall y \forall z (F(x, y) \wedge F(x, z) \wedge (y \neq z) \rightarrow \neg F(y, z))$

Nested Quantifier

Exercise: translate the following English sentence into logical expression

"There is a rational number in between every pair of distinct rational numbers"

Use predicate $Q(x)$, which is true when x is a rational number

$$\forall x, y (Q(x) \wedge Q(y) \wedge (x < y) \rightarrow \exists u (Q(u) \wedge (x < u) \wedge (u < y)))$$

Rules of Inference

Rules of inference provide the justification of the steps used in a proof.

One important rule is called **modus ponens** or the **law of detachment**. It is based on the tautology $(p \wedge (p \rightarrow q)) \rightarrow q$. We write it in the following way:

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

The two **hypotheses** p and $p \rightarrow q$ are written in a column, and the **conclusion** below a bar, where \therefore means "therefore".

Rules of Inference

The general form of a rule of inference is:

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

The rule states that if p_1 and p_2 and ... and p_n are all true, then q is true as well.

Each rule is an established tautology of
$$p_1 \wedge p_2 \wedge \dots \wedge p_n \rightarrow q$$

These rules of inference can be used in any mathematical argument and do not require any proof.

Rules of Inference

$$\frac{p}{\therefore p \vee q}$$

Addition

$$\frac{p \wedge q}{\therefore p}$$

Simplification

$$\frac{p \quad q}{\therefore p \wedge q}$$

Conjunction

$$\frac{\neg q \quad p \rightarrow q}{\therefore \neg p}$$

Modus
tollens

$$\frac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$$

Hypothetical
syllogism
(chaining)

$$\frac{r \quad p \vee q \quad \neg p}{\therefore q}$$

Disjunctive
syllogism
(resolution)

Arguments

Just like a rule of inference, an **argument** consists of one or more hypotheses (or premises) and a conclusion.

We say that an argument is **valid**, if whenever all its hypotheses are true, its conclusion is also true.

However, if any hypothesis is false, even a valid argument can lead to an incorrect conclusion.

Proof: show that **hypotheses** \rightarrow **conclusion** is true using rules of inference

Arguments

Example:

"If 101 is divisible by 3, then 101^2 is divisible by 9.
101 is divisible by 3. Consequently, 101^2 is divisible by 9."

Although the argument is **valid**, its conclusion is **incorrect**, because one of the hypotheses is false ("101 is divisible by 3").

If in the above argument we replace 101 with 102, we could correctly conclude that 102^2 is divisible by 9.

Arguments

Which rule of inference was used in the last argument?

p: "101 is divisible by 3."

q: "101² is divisible by 9."

$$\begin{array}{lcl} p & & \\ p \rightarrow q & \text{Modus} & \\ \hline \therefore q & \text{ponens} & \end{array}$$

Unfortunately, one of the hypotheses (p) is false. Therefore, the conclusion q is incorrect.

Arguments

Another example:

"If it rains today, then we will not have a barbeque today. If we do not have a barbeque today, then we will have a barbeque tomorrow.
Therefore, if it rains today, then we will have a barbeque tomorrow."

This is a **valid** argument: If its hypotheses are true, then its conclusion is also true.

Arguments

Let us formalize the previous argument:

p: "It is raining today."

q: "We will not have a barbecue today."

r: "We will have a barbecue tomorrow."

So the argument is of the following form:

$$\begin{array}{ll} p \rightarrow q & \\ q \rightarrow r & \text{Hypothetical} \\ \hline \therefore p \rightarrow r & \text{syllogism} \end{array}$$

Arguments

Another example:

Gary is either intelligent or a good actor.
If Gary is intelligent, then he can count
from 1 to 10.

Gary can only count from 1 to 3.
Therefore, Gary is a good actor.

i: "Gary is intelligent."

a: "Gary is a good actor."

c: "Gary can count from 1 to 10."

Arguments

i: "Gary is intelligent."

a: "Gary is a good actor."

c: "Gary can count from 1 to 10."

Step 1: $\neg c$ Hypothesis

Step 2: $i \rightarrow c$ Hypothesis

Step 3: $\neg i$ Modus tollens Steps 1 & 2

Step 4: $a \vee i$ Hypothesis

Step 5: a Disjunctive Syllogism
Steps 3 & 4

Conclusion: a ("Gary is a good actor.")

Arguments

Yet another example:

If you listen to me, you will pass CS 320.

You passed CS 320.

Therefore, you have listened to me.

Is this argument valid?

No, it assumes $((p \rightarrow q) \wedge q) \rightarrow p$.

This statement is not a tautology. It is false if p is false and q is true.

Rules of Inference for Quantified Statements

$\forall x P(x)$

$\therefore P(c)$ if

$c \in U$
 $P(c)$ for an arbitrary $c \in U$

$\therefore \forall x P(x)$

Universal
instantiation

Universal
generalization

$\exists x P(x)$

$\therefore P(c)$ for some element $c \in U$

Existential
instantiation

$P(c)$ for some element $c \in U$

$\therefore \exists x P(x)$

Existential
generalization

Rules of Inference for Quantified Statements

Example:

Every UMB student is a genius.

George is a UMB student.

Therefore, George is a genius.

$U(x)$: "x is a UMB student."

$G(x)$: "x is a genius."

Rules of Inference for Quantified Statements

The following steps are used in the argument:

Step 1: $\forall x (U(x) \rightarrow G(x))$ Hypothesis

Step 2: $U(\text{George}) \rightarrow G(\text{George})$ Univ. instantiation
using Step 1

Step 3: $U(\text{George})$ Hypothesis

Step 4: $G(\text{George})$ Modus ponens
using Steps 2 & 3

$$\frac{\forall x P(x)}{\therefore P(c) \text{ if } c \in U} \quad \begin{array}{l} \text{Universal} \\ \text{instantiation} \end{array}$$

Proving Theorems

Direct proof:

An implication $p \rightarrow q$ can be proved by showing that if p is true, then q is also true.

Example: Give a direct proof of the theorem "If n is odd, then n^2 is odd."

Idea: Assume that the hypothesis of this implication is true (n is odd). Then use rules of inference and known theorems of math to show that q must also be true (n^2 is odd).

Proving Theorems

 n is odd.

Then $n = 2k + 1$, where k is an integer.

$$\begin{aligned}\text{Consequently, } n^2 &= (2k + 1)^2. \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1\end{aligned}$$

Since n^2 can be written in this form, it is odd.

Proving Theorems

Indirect proof:

An implication $p \rightarrow q$ is equivalent to its **contra-positive** $\neg q \rightarrow \neg p$. Therefore, we can prove $p \rightarrow q$ by showing that whenever q is false, then p is also false.

Example: Give an indirect proof of the theorem "If $3n + 2$ is odd, then n is odd."

Idea: Assume that the conclusion of this implication is false (n is even). Then use rules of inference and known theorems to show that p must also be false ($3n + 2$ is even).

Proving Theorems

n is even.

Then $n = 2k$, where k is an integer.

$$\begin{aligned}\text{It follows that } 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1)\end{aligned}$$

Therefore, $3n + 2$ is even.

We have shown that the contrapositive of the implication is true, so the implication itself is also true (If $3n + 2$ is odd, then n is odd).

Proving Theorems

Indirect Proof is a special case of proof by contradiction

Suppose n is even (negation of the conclusion).

Then $n = 2k$, where k is an integer.

$$\begin{aligned}\text{It follows that } 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1)\end{aligned}$$

Therefore, $3n + 2$ is even.

However, this is a contradiction since $3n + 2$ is given to be odd, so the conclusion (n is odd) holds.

Another Example on Proof

Anyone performs well is either intelligent or a good actor.

If someone is intelligent, then he/she can count from 1 to 10.

Gary performs well.

Gary can only count from 1 to 3.

Therefore, not everyone is both intelligent and a good actor

$P(x)$: x performs well

$I(x)$: x is intelligent

$A(x)$: x is a good actor

$C(x)$: x can count from 1 to 10

Another Example on Proof

Hypotheses:

1. Anyone performs well is either intelligent or a good actor.

$$\forall x (P(x) \rightarrow I(x) \vee A(x))$$

2. If someone is intelligent, then he/she can count from 1 to 10.

$$\forall x (I(x) \rightarrow C(x))$$

3. Gary performs well.

$$P(G)$$

4. Gary can only count from 1 to 3.

$$\neg C(G)$$

Conclusion: not everyone is both intelligent and a good actor

$$\neg \forall x (I(x) \wedge A(x))$$

Another Example on Proof

Direct proof:

- Step 1: $\forall x (P(x) \rightarrow I(x) \vee A(x))$ Hypothesis
Step 2: $P(G) \rightarrow I(G) \vee A(G)$ Univ. Inst. Step 1
Step 3: $P(G)$ Hypothesis
Step 4: $I(G) \vee A(G)$ Modus ponens Steps 2 & 3
Step 5: $\forall x (I(x) \rightarrow C(x))$ Hypothesis
Step 6: $I(G) \rightarrow C(G)$ Univ. inst. Step 5
Step 7: $\neg C(G)$ Hypothesis
Step 8: $\neg I(G)$ Modus tollens Steps 6 & 7
Step 9: $\neg I(G) \vee \neg A(G)$ Addition Step 8
Step 10: $\neg(I(G) \wedge A(G))$ Equivalence Step 9
Step 11: $\exists x \neg(I(x) \wedge A(x))$ Exist. general. Step 10
Step 12: $\neg \forall x (I(x) \wedge A(x))$ Equivalence Step 11

Conclusion: $\neg \forall x (I(x) \wedge A(x))$, not everyone is both intelligent and a good actor.

Summary, Section 1.5

- Terminology (axiom, theorem, conjecture, argument, etc.)
- Rules of inference (Tables 1 and 2)
- Valid argument (hypotheses and conclusion)
- Construction of valid argument using rules of inference
 - For each rule used, write down and the statements involved in the proof
- Direct and indirect proofs
 - Other proof methods (e.g., induction, pigeon hole) will be introduced in later chapters