Graphs

Definition: Graph

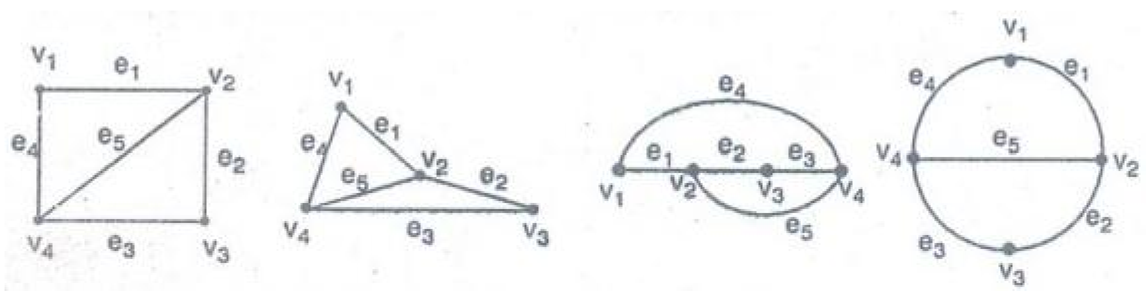
A graph G = (V(G), E (G)) consists of V, a non-empty set of vertices (nodes or points) and E, a set of edges (also called lines).

i.e., A graph G is an ordered triple (V (G), E (G), ϕ) consists of a non-empty set V called the set of vertices (nodes or points) of the graph G, E is said to be the set of edges of the graph G, and is a mapping from the set of edges E to a set of order or un ordered pairs of elements of V.

Example :

Let G = (V (G), E (G), ϕ) where V (G)= $\{v_1, v_2, v_3, v_4\}$ E (G)=(e_1, e_2, e_3, e_4, e_5) and ϕ is defined by $\phi(e_1) = \{v_1, v_2\}$, $\phi(e_2) = \{v_2, v_3\}$, $\phi(e_3) = \{v_3, v_4\}$, $\phi(e_4) = \{v_4, v_1\}$, $\phi(e_5) = \{v_1, v_3\}$

Now the diagramatic form of G is



It should be noted that, in drawing a graph, it is immaterial whether the edges are drawn straight or curved, long or short, the important point is how the vertices are joined up.

The above graphs are same.

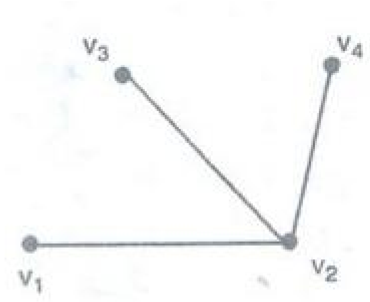
Note: 1. We denote the graph G as G (V, E) or simply as G.

2. If e ϵ E is an edge and $\phi(e) = \{v_1, v_2\}$, then we say that e is an edge joining v_1 and v_2, and the vertices v_1 and v_2 are called the ends (end vertices) of e.

3. If graphs, an edge should not pass through any points (vertices) other than the two end vertices of the edge.

Definition: Adjacent vertices

Any pair of vertices which are connected by an edge in a graph is called adjacent vertices.

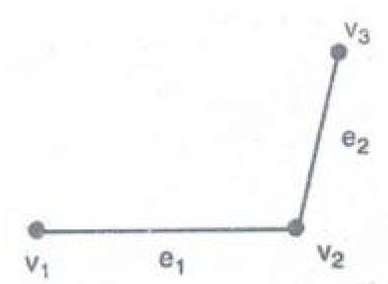


Here v_1, v_2 ; v_2, v_4 ; v_2, v_3 are adjacent vertices

v_1, v_3 ; v_3, v_4 ; v_1 v_4 are not adjacent.

Definition: Adjacent edges:

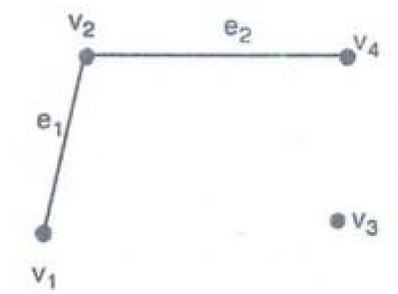
If two distinct edges are incident with a common vertex then they are called adjacent edges.



Here e_1 and e_2 are incident with a common vertex v_2.

Definition: Isolated vertex

In any graph, a vertex which is not adjacent to any other vertex is called an isolated vertex. Otherwise the vertex has no incident edge.



Here v_3 has no incident edge. Therefore the vertex v_3 is called isolated vertex.

Note :

1. A graph with p vertices and q edges is called a (p, q) graph.

2. The graph (p, 0) is trivial or null graph.

3. If any two edges are intersected then their intersection is not considered as a vertex.

4. The set of edges in a null graph is empty.

Definition: Label graph

A graph in which each vertex is assigned a unique name or label is called a label graph.

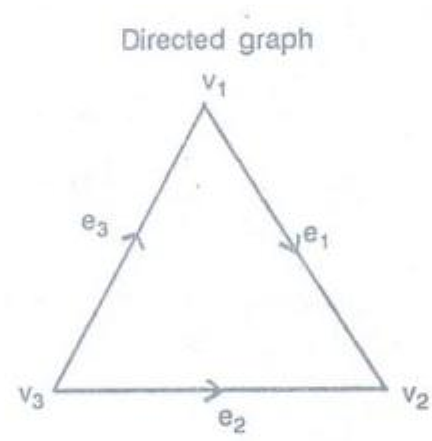
Definition: Directed graph and undirected graph

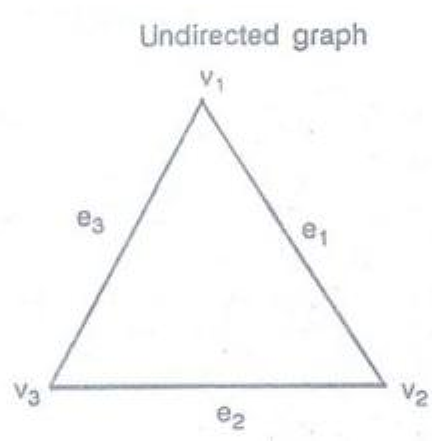
In a graph G (V, E), an edge which is associated with an ordered pair of vertices is called a directed edge of graph G, while an edge which is associated with an unordered pair of vertices is called an undirected edge.

A graph in which every edge is directed is called a directed graph simply a digraph.

A graph in which every edge is undirected is called an undirected graph.

The end vertices of an edge are said to be incident with the edge and vice versa.



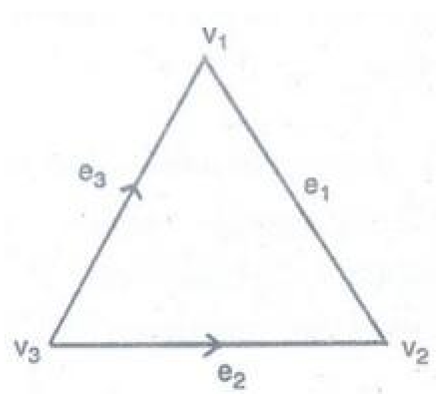


The edge e_1 is incident with the vertices v_1 and v_2 also the vertex v_1 is incident with e_1 and e_3.

The vertices v_1 and v_2 are also called the initial and terminal vertices of the edge e_1.

Definition: Mixed graph

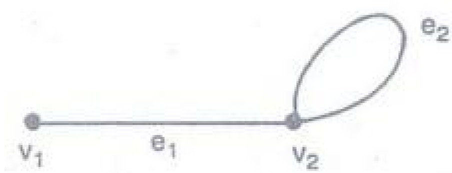
If some edges are directed and some are undirected in a graph, then the graph is a mixed graph.

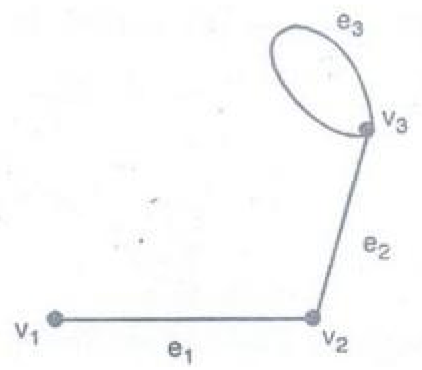


Definition: Loop

A loop is an edge whose vertices are equal.

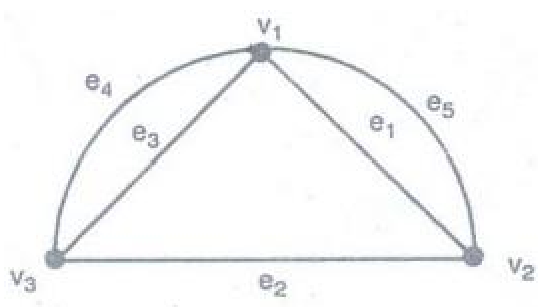
i.e., An edge of a graph which joins a vertex to itself is called a loop.





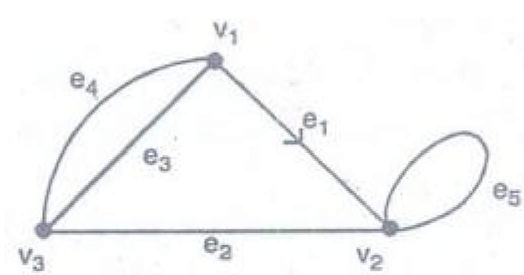
Definition: Parallel edges (Multiple edges)

Multiple edges are edges having the same pair of vertices.



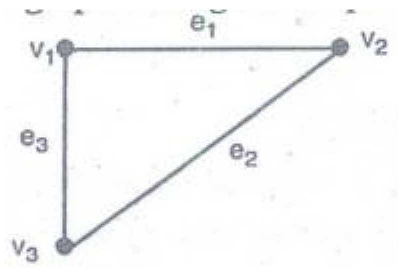
Definition: Multi graph

Any graph which contains some parallel edges and loops is called as multi graph.



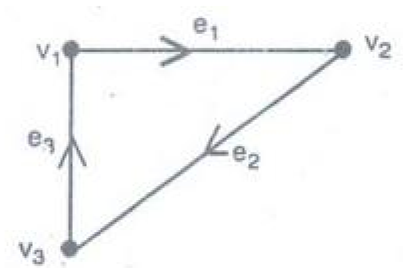
Definition: Simple graph

A simple graph is a graph having no loops or multiple edges.



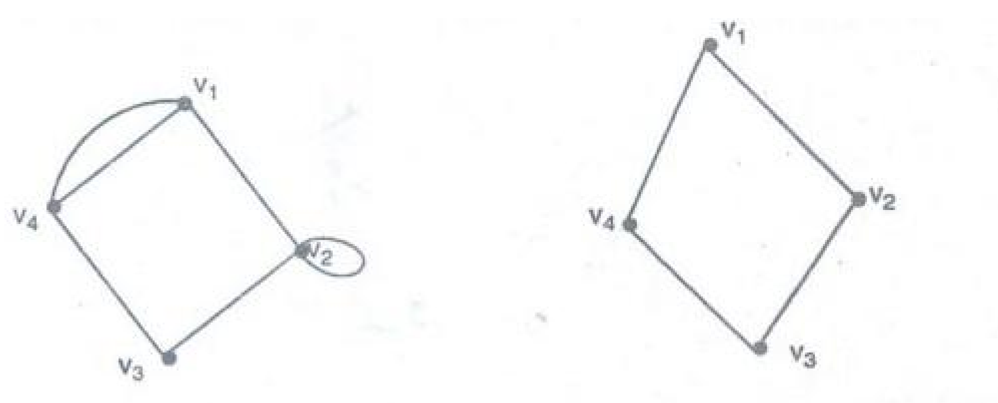
Definition: Simple directed graph

When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph.



Definition: Underlying simple graph

A graph obtained by deleting all loops and parallel edges from a graph is called underlying simple graph.



Multigraph G Underlying simple graph of G.

Definition : Finite graph

A graph G is finite if and only if both the vertex set V(G) and the edge set E (G) are finite, otherwise the graph is infinite.

Example : Let V(G) = Z and E (G) = {eij / |i-j| = 1} clearly, the graph G is infinite.

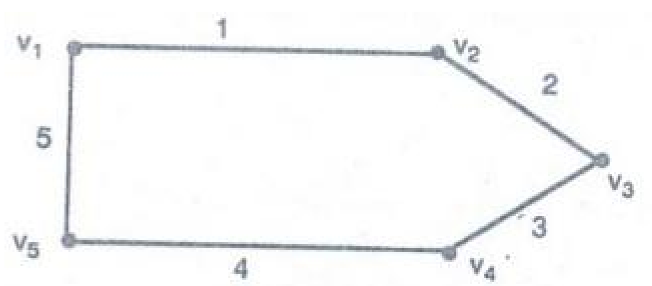
Note: Hereafter, a graph means that is a finite graph unless otherwise stated.

Definition: Multiplicity m.

When there are m directed edges, each associated to an ordered pair of vertices (*u, v*), we say that (*u, v*) is an edge of multiplicity m.

Definition: Weighted graph

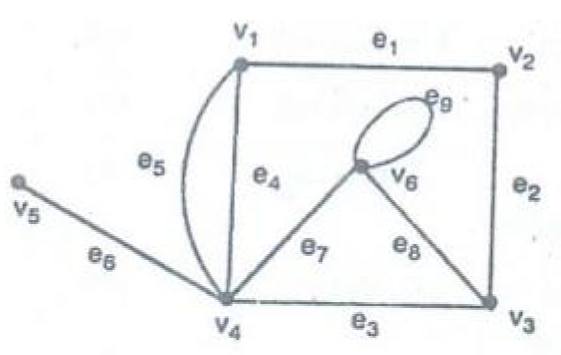
A graph in which weights are assigned to every edge is called a weighted graph.



here 1, 2, 3, 4, 5 are weights assigned to each edge respectively.

Note :

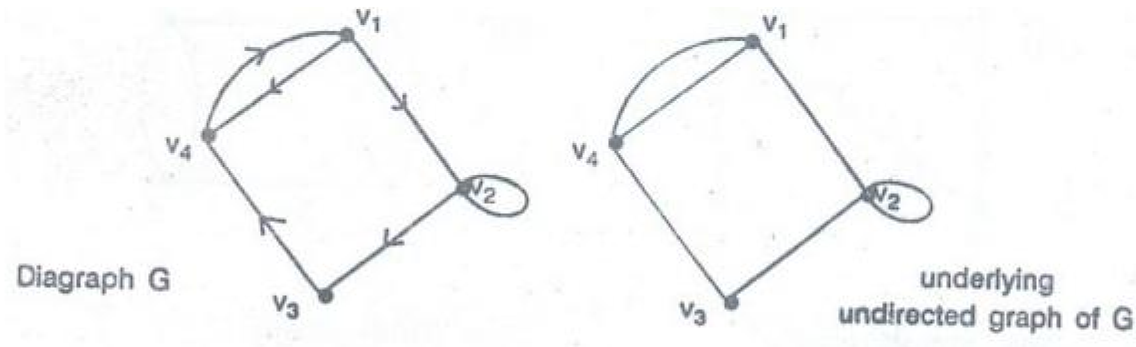
If the graph G is finite, |V| denotes the number of vertices of G known as order of G and |E| denotes the number of edges of G, known as size of G,



For this graph |V| = 6; |E| = 9

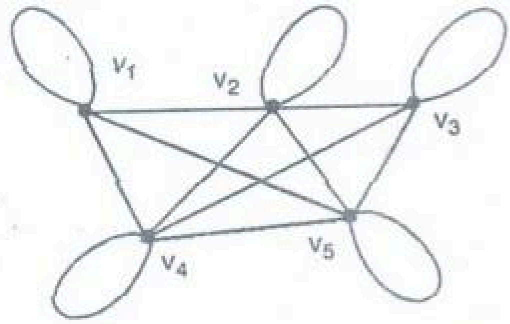
Definition: Underlying undirected graph

A graph obtained by ignoring the direction of edges in a directed graph is called underlying undirected graph.



Definition Pseudographs :

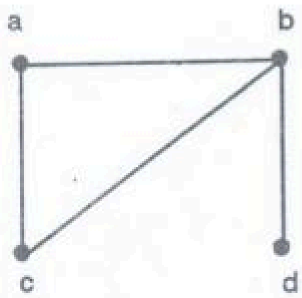
Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices, are sometimes called Pseudographs.



S.No.	Type	Edges	Multiple Edges	Loops
1.	Simple graph	Undirected	No	No
2.	Multigraph	Undirected	Yes	No
3.	*Pseudograph*	Undirected	Yes	Yes
4.	Simple *directed graph*	Directed	No	No
5.	Directed *multigraph*	Directed	Yes	Yes
6.	Mixed graph	Directed and *undirected*	Yes	Yes

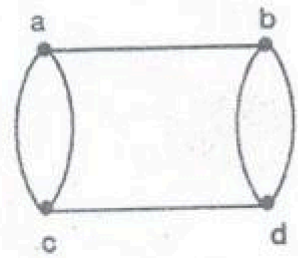
Example 1. What type of graph the following are

(a)



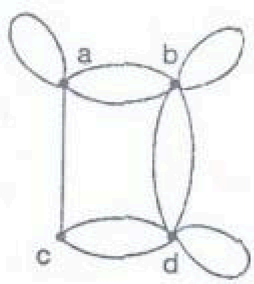
Ans. Simple graph

(b)



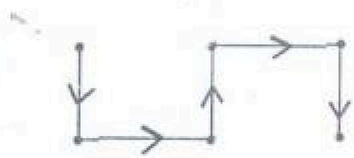
Ans. Multi graph

(c)



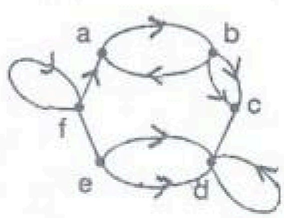
Ans. Pseudo graph

(d)



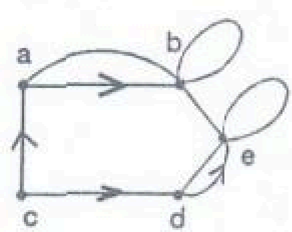
Ans. Simple directed graph

(e)



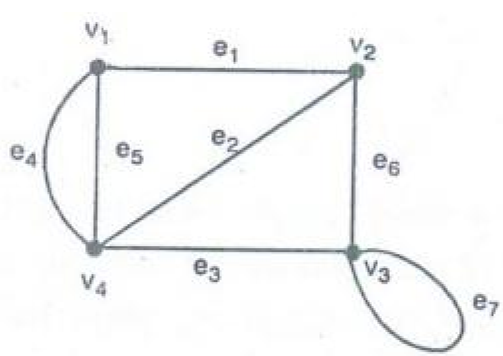
Ans. Directed multi graph

(f)



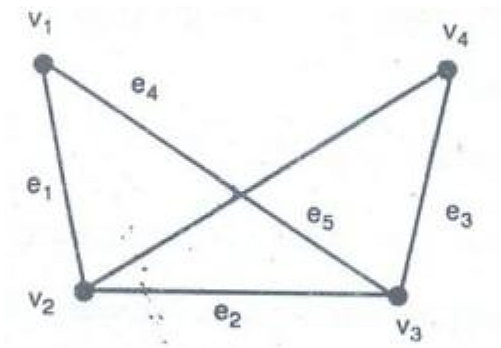
Ans. Mixed graph

Example 2. The diagram shows a multigraph G, Why G is not a simple graph ?



Solution: G is not a simple graph since it contains multiple edges e_4, e_5 also a loop e_7.

Example 3. Describe formally the graph given below :



Solution : G = (V,E)

V = $\{v_1, v_2, v_3, v_4\}$

E = $\{e_1, e_2, e_3, e_4, e_5\}$

E (a) = $\{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_1, v_3), (v_2, v_4)\}$

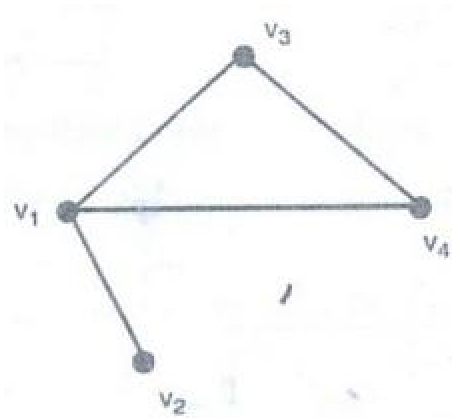
Example 4. Draw a diagram for the following graph

G = G(V,E)

V = $\{v_1, v_2, v_3, v_4\}$

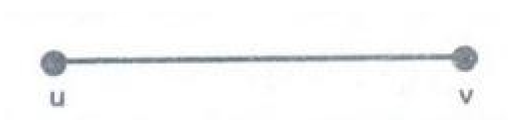
E = $\{(v_1, v_2), (v_4, v_1), (v_3, v_1), (v_3, v_4)\}$

Solution :



Example 5. Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to {u, v} is a symmetric, irreflexive relation on G.

Solution :



Given: "G is a simple graph, R is a relation to (u, v)

(i) To prove symmetric

i.e., to prove uR$v \Rightarrow v$Ru

If u R v, then there is an edge associated with $\{u, v\}$

But $\{u,v\} = \{v, u\}$ so this edge is associated with $\{v, u\}$

$\Rightarrow v\ R\ u$

(ii) To prove irreflexive :

A simple graph doesnot allow loops,

$u\ R\ u$ never holds.

Irreflexive.

Graph Terminology and Special Types of Graphs

Definition: Two vertices *u* and *v* in an undirected graph G are called adjacent (or neighbors) in G if *u, v* are endpoints of an edge of G.

If e is associated with (*u, v*), the edge *e* is called incident with the vertices *u* and *v*. The edge *e* is also said to connect *u* and *v*.

The vertices *u* and *v* are called endpoints of an edge associated with (*u, v*).

Definition: The degree of a vertex :

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

The degree of the vertex is denoted by deg( ).

Example :

Let v be a vertex in a graph G. Then the degree $d_G(V)$ of the incident with *v* (each loop is counted twice). The $d_G(V)$ can also be denoted by deg G(V) (or explicity, we use d (V) or deg (V) to denote the degree of V).

deg (V_1) = 6

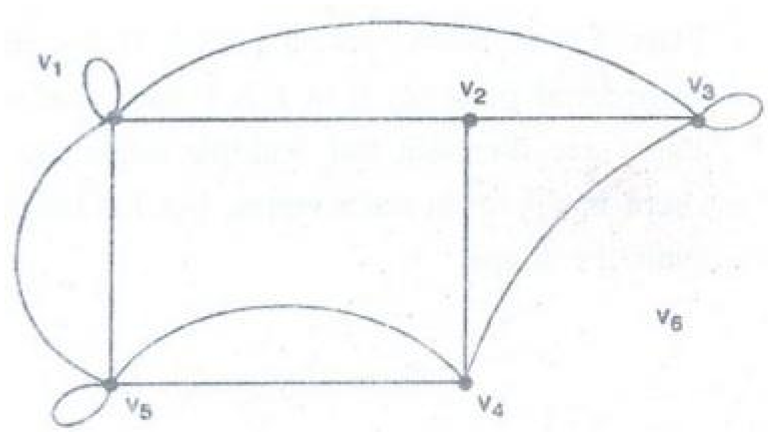
deg (V_2) = 3

deg (V_3) = 5

deg (V_4) = 4

deg (V_5) = 6

deg (V_6) = 0



Note :

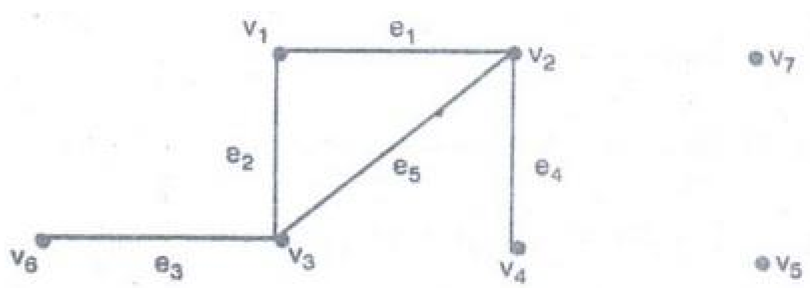
1. Let G be an undirected graph with |E| edges and |V| = n vertices, then $\Sigma_{ni=1}$ deg (v_i) = 2 |E|.

2. In any graph, the number of vertices of odd degree is even.

3. A vertex of degree one is called a pendant or end vertex in G.

4. A vertex of degree zero is called an isolated vertex in G.

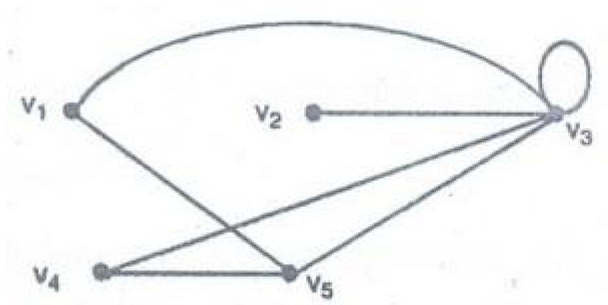
5. Two adjacent edges are said to be in series if their common vertex is of degree two.



The vertices V_4, V_6 are pendant vertices.

The vertices V_5, V_7 are isolated vertices.

Example 1. What are the degrees of the vertices in the graph G.



Solution :

deg (v_1) = 2, deg (v_2) = 1, deg (v_3) = 6

deg (v_4) = 2, deg (v_5) = 3.

Example 2. How many edges are there in a graph with 10 vertices each of degree six ?

Solution: Sum of the degrees of the 10 vertices

is (6) (10) = 60

i.e., 2e = 60

e = 30

Example 3. Show that the sum of degree of all the vertices in a graph G, is even.

Proof : Each edge contribute two degrees in a graph.

Also, each edge contributes one degree to each of the vertices on which it is incident.

Hence, if there are N edges in G, then

2N = d (v_1) + d (v_2) + …. + d (v_N)

Thus, 2 N is always even.

[The Handshaking Theorem] [A.U N/D 2012]

Theorem: For any graph G with E edges and V vertices

$v1$, v_2, …., v_n, $\Sigma_{ni=1}$ d (v_i) = 2 E

Proof :

Let G = G (V,E) be any graph, where V = {v_1, v_2, …, v_n} and E = (e_1, e_2,..., e_n}

Since, each edges contributes twice as a degree, the sum of the degree of all vertices in G is twice as the number of edges in G.

i.c., $\Sigma_{ni=1}$ d (v_i) = 2 |E| = 2e

Note: This theorem applies even if multiple edges and loops are present.

Theorem: The number of odd degree vertices is always even.

Let G = {V, E} be any graph with 'n' number of vertices and '*e*' number of edges.

Let $V_1, 2, \ldots, V_k$ be the vertices of odd degree and $V_1', V_2', \ldots, V_m'$ be the vertices of even degree.

To prove, k is even

We know that $\sum_{i=1}^{n} d(V_i) = 2|E| = 2e$

$$\Rightarrow \sum_{i=1}^{k} d(V_i) + \sum_{j=1}^{m} d(V_j') = 2e$$

Each of $d(V_j)$ is even $\Rightarrow \sum_{j=1}^{m} d(V_j')$ and $2e$

are even numbers (being the sum of even numbers)

$$\therefore \sum_{i=1}^{k} d(V_i) + \text{ an even number } = \text{ an even number}$$

$$\Rightarrow \sum_{i=1}^{k} d(V_i) = \text{ an even number.}$$

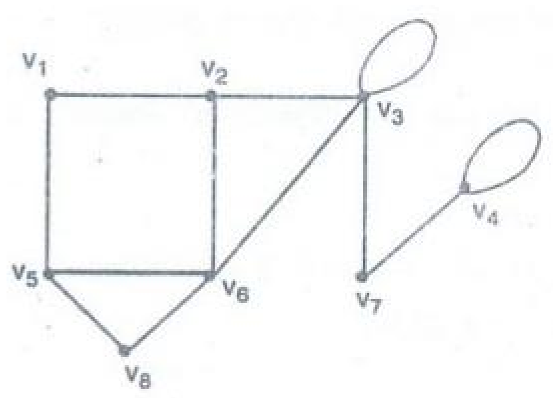
Since, each term d (V_i) is odd.

Therefore, the number of teams in the LHS sum must be even.

⇒ K is even.

Hence, the theorem.

Example 4. Verify that the sum of the degree of all the vertices is even for the graph.



Solution: The sum of degree of all the vertices is

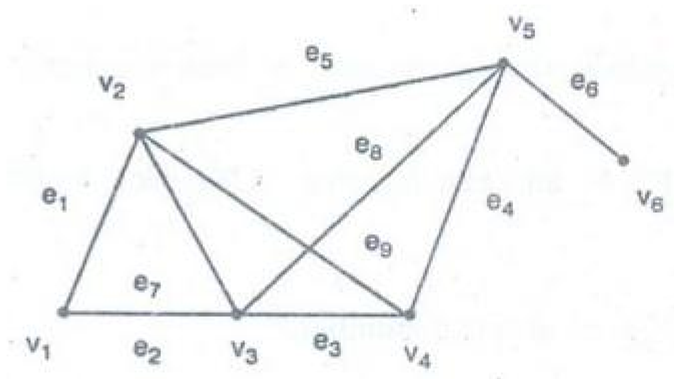
= d (v_1) + d (v_2) + d (v_3) + d (v_4) + d (v_5) + d (v_6) + d (v_7) + d (v_8)

= 2 + 3 + 5 + 3 + 3 + 4 + 2 + 2

= 24 which is even.

Note: Odd vertices means vertices of odd degree.

Example 5. Verify the handshaking theorem for the graph.



Solution: To prove Σ deg (v_i) = (2) (no. of edges)

i.e., Σ deg (v_i) = (2) (9) = 18

Σ deg (v_i) = deg (v_1) + deg (v_2) + deg (v_3) + deg (v_4) + deg (v_5) + deg (v_6)

= 2 + 4 + 4 + 3 + 4 + 1

= 18

Hence the theorem is true.

Theorem: A simple graph with at least two vertices has at least two vertices of same degree.

Proof: Let G be a simple graph with n ≥ 2 vertices.

The graph G has no loop and parallel edges.

Hence the degree of each vertices is ≤ n - 1.

Suppose that all the vertices of G are of different degrees.

Following degrees 0, 1, 2, 3, …, n - 1 are possible for n vertices of G.

Let *u* be the vertex with degree 0. Then u is an isolated vertex.

Let *v* be the vertex with degree n - 1 then v has n - 1 adjacent vertices.

Because *v* is not an adjacent vertex of itself, therefore every vertex of G other than *u* is an adjacent vertex of G other than *u* is an adjacent vertex *u*.

Hence u cannot be an isolated vertex, this contradiction proves that a simple graph contains two vertices of same degree.

Note: The converse of the above theorem is not true.

Example 6. Show that the degree of a vertex of a simple graph G on n vertices cannot exceed n-1.

Solution: Let v be a vertex of G because G is simple, no multiple edges or loops are allowed in G. Thus v can be adjacent to at most all the remaining n - 1 vertices of G.

Hence v may have maximum degree n - 1 in G.

Then $0 \leq \deg G(v) \leq n - 1 \ \forall \ v \in V(G)$

Example 7. Show that in a group, there must be two people who know the same number of other people in the group.

Solution: Construct the simple graph model in which V is the set of people in the group and there is an edge associated with (u, v) if u and v know each other. Then the degree of vertex v is the number of people v knows.

We know that there are two vertices with the same degree. Therefore there are two people who know the same number of other people in the group.

Example 8. Is there a simple graph corresponding to the following degree sequences? (i) (1, 1, 2, 3) (ii) (2, 2, 4, 6)

Solution :

(i) There are odd number (3) of odd degree vertices, 1, 1 and 3. Hence there exist no graph corresponding to this degree sequence.

(ii) Number of vertices in the graph sequence is four and the maximum degree of a vertex is 6 which is not possible as the maximum degree cannot exist one less than the number of vertices.

Example 9. Show that the maximum number of edges in a simple graph with n vertices is n (n - 1) / 2

Solution: Use handshaking theorem.

i.e,. $\Sigma_{i=1}$ d (v_i) 2 e,

where *e* is the number of edges with n vertices in the graph G.

i.e., d (v_1) + *d* (v_2) + ... *d* (v_n) = 2*e* ….. (1)

Since we know that the maximum degree of each vertex in the graph G can be (n - 1)

(1) ⇒ (n − 1) + (n − 1) + ... to n terms = 2*e*

⇒ n (n − 1) = 2*e*

e = n (n - 1) / 2

Hence the maximum number of edges in any simple graph with n vertices is n (n - 1) / 2

Definition: When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u.

The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v).

The initial vertex and terminal vertex of a loop are the same.

Definition: In a graph with directed edges the in-degree of a vertex v, denoted by deg^- (*v*), is the number of edges with v as their terminal vertex.

The out-degree of with v, denoted by deg^+ (*v*), is the number of edges with *v* as their initial vertex.

Note: A loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

In degree

In a directed graph G, the in degree of v denoted by in deg G (v) or deg^{-1}_G (v), is the number of edges ending at *v*.

Out degree

In a directed graph G, the out degree of vertex of v of G denoted by out deg_G(*v*) or deg^+_G (*v*), is the number of edges beginning at *v*.

Note:

(1) The sum of the in degree and out degree of a vertex is called the total degree of the vertex.

(2) A vertex with zero in degree is called a source and a vertex with zero out degree is called a sink.

Theorem: If G = (V, E) be a directed graph with e edges, then

$\Sigma_{v \epsilon V} \deg_{+G}(v) = \Sigma_{v \epsilon V} \deg_{-G}(v) = e$

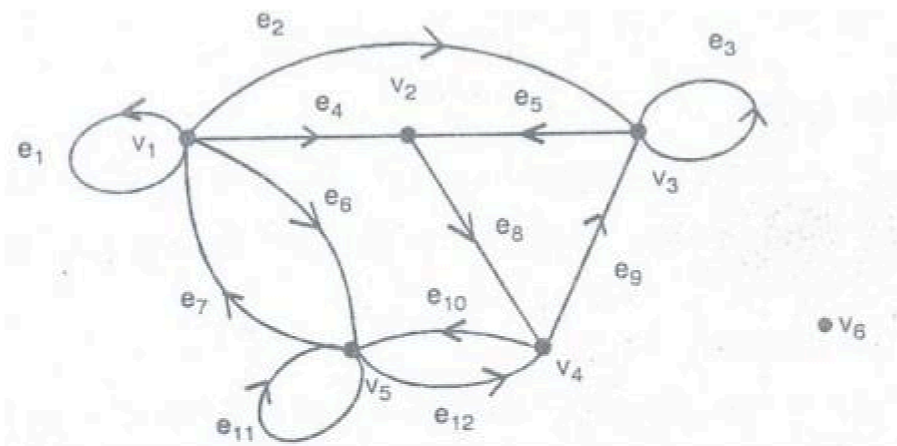
i.e., the sum of the outdegrees of the vertices of a diagraph G equals the sum of in degrees of the vertices which equals the number of edges in G.

Proof: Each edge has an initial vertex and a terminal vertex.

⇒ Each edge contributes one out degree to its initial vertex and one indegree to its terminal vertex.

Thus the sum of the indegrees and the sum of the out degrees of all vertices in a directed graph are same.

Example 10. Verify $\Sigma_{ni=1} \deg_{+G}(v_i) = \Sigma_{ni=1} \deg_{-}(v_i) = |E| = e$ in the following graph.



Solution :

deg	out degree : deg^-	indegree deg^+
v_1	4	2
v_2	1	2
v_3	2	3
v_4	2	2
v_5	3	9
v_6	0	0
Sum	12	12

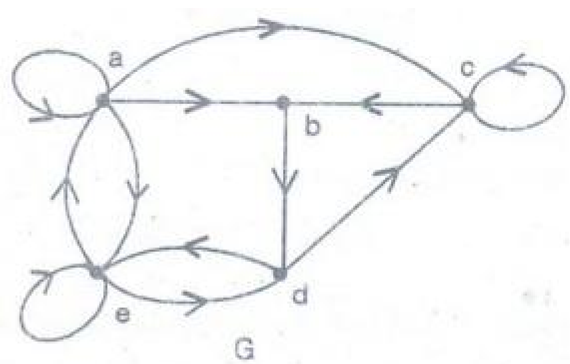
$$\therefore \sum_{i=1}^{n} deg^+(v_i) = \sum_{i=1}^{n} deg^-(v_i) = e = 12$$

Example 11. What do the in-degree and the out-degree of a vertex in a directed graph modeling a round-robin tournament represent?

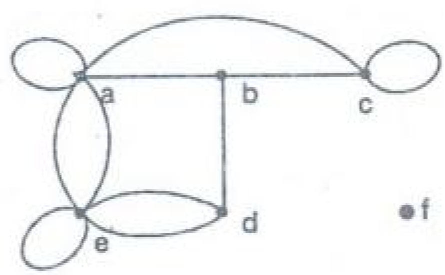
Solution: (deg_+ (v), deg_- (v)) is the win-loss record of v.

Definition: Underlying undirected graph. The undirected graph that results from ignoring directions of edges is called the underlying undirected graph.

Example 12. Construct the underlying undirected graph for the graph with directed edges in the following figure.



Solution :



The minimum of all the degrees of the vertices of a graph G is denoted by δ (G), and the maximum of all the degrees of the vertices of G is denoted by Δ (G).

In otherwords

$\delta (G) = \min \{deg (V) / V \epsilon V (G)\}$ and

$\Delta (G) = \max \{deg (V) / V \epsilon V (G)\}$

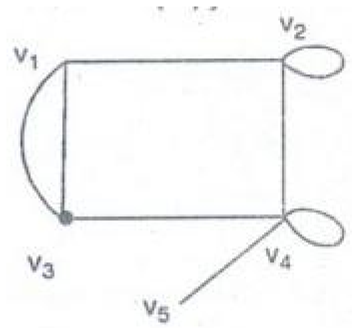
$d (V_1) = d (V_3) = 3$

$d (V_2) = d(V_4) = 4$

$d (V_5) = 1$

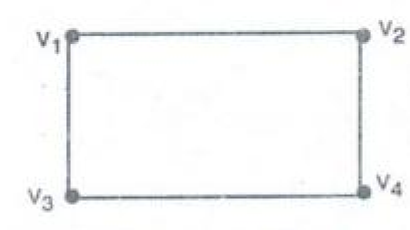
$\delta (G) = 1$

$\Delta (G) = 4$



Clearly δ (G) $\leq$ Δ (G) = 4

If δ (G) = Δ (G) = K i.e., if each vertex of a graph G has degree K, then G is said to be K - regular graph of degree K.

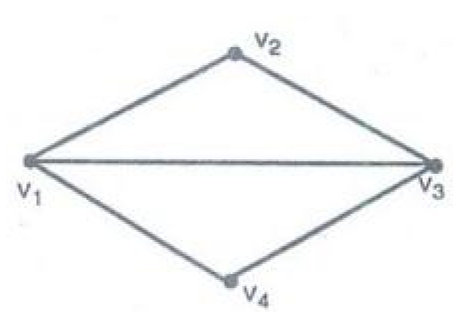


d (V_1) = d (V_2) = d (V_3) = d (V_4) = 2

The given graph is 2-regular (or) regular graph of degree 2.

If G is a K-regular graph, then $\delta = 2\,|E|\,/\,|V| = \Delta$

If V_1, V_2, V_3, …. V_n are the n vertices of G, then the sequence (d_1, d_2, …, d_n), where d_i = deg (v_i) is the degree sequence of G.

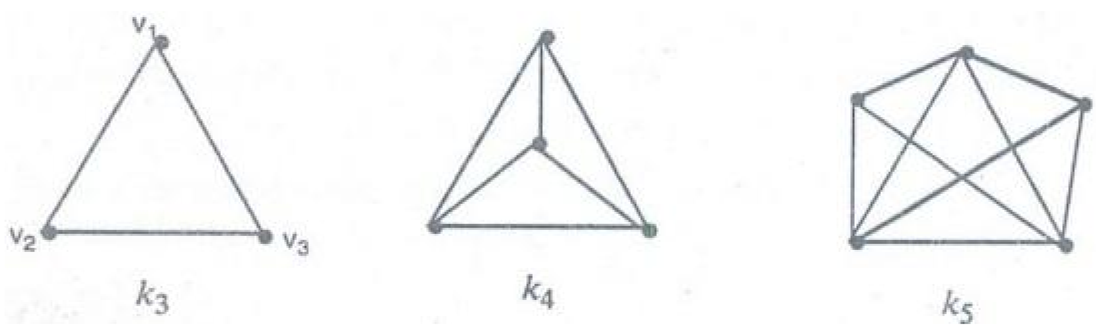


A degree sequence of G: (2, 2, 3, 5)

A degree sequence d = (d_1, d_2, …. d_n) is graphic if there is a simple undirected graph with degree sequence d.

Is there a graph with degree sequence (1, 3, 3, 3, 7, 6, 6). No, because the no. of vertices with odd degree is odd, a contradiction to corollary; the number of vertices of odd degree must be even.

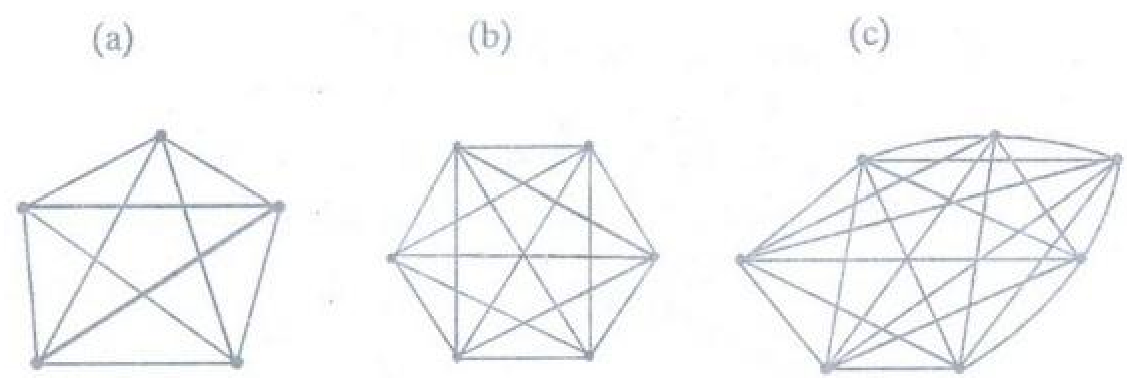
A simple graph in which every pair of distinct vertices are adjacent is called a complete graph. If G has n vertices then the complete graph will be denoted by K_n.



Example 13. Draw these graphs

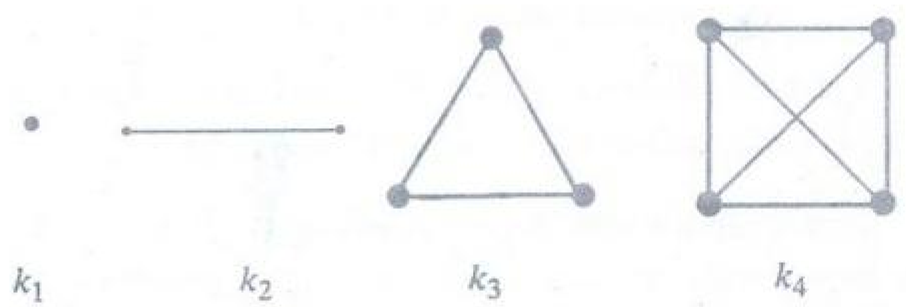
(a) K_5 (b) K_6 (c) K_7

Solution :



Example 14. Draw Graphs K_n for $1 \leq n \leq 4$

Solution :



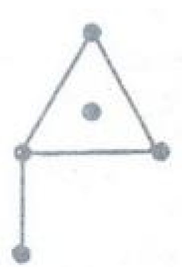
Example 15. What is the degree sequence of K, where n is a positive integer? Explain your answer.

Solution: Each of the n vertices is adjacent to each of the other n-1 vertices, so the degree sequence is n - 1, n - 1, ..., n - 1 (n terms)

Example 16. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence. (a) 5, 4, 3, 2, 1 (b) 3, 2, 2, 1, 0 (c) 1, 1, 1, 1, 1

Solution :

(a) No, (5+4+3+2+1=15) sum of degree is odd

(b) Yes 

(c) No, (1+1+1+1+1=5) sum of degrees is odd.

Example 17. How many vertices and how many edges of K_n?

Ans. n vertices, n (n - 1)/2 edges.

Example 18. Find the degree sequence of each of the following graphs (a) K_4 (b) K_5 (c) K_2

Solution :

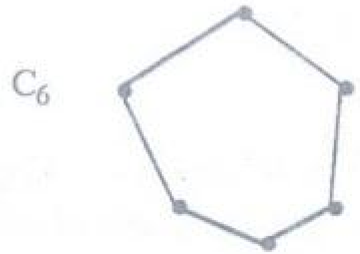
(a) 3, 3, 3, 3

(b) 4, 4, 4, 4, 4

(c) 1, 1

Definition: Cycle Graph

A cycle graph of order 'n' is a connected graph whose edges form a cycle of length 'n' and denoted by C_n.



Note:

(1) In a graph a cycle that is not a loop must have length atleast 3, but there may be cycles of length 2 in a multigraph.

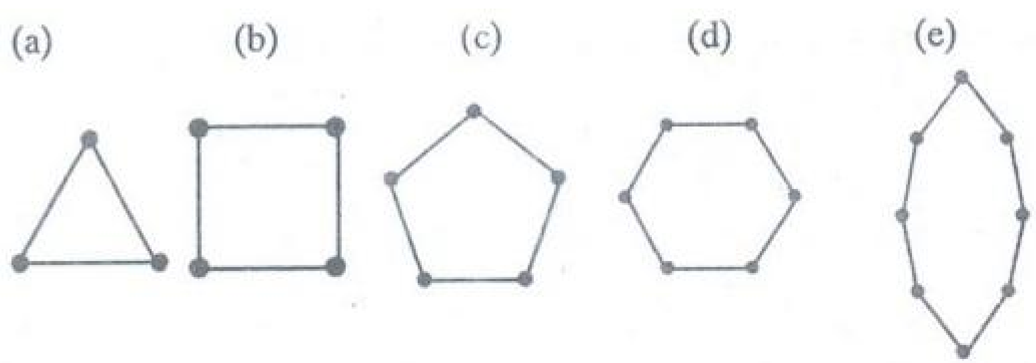
(2) A simple digraph having no cycles is called a cyclic graph.

(3) An cyclic graph cannot have any loops.

(4) The cycle C_n, $n \geq 3$, consists of n vertices 1, 2, …, n and edges {1, 2}, {2, 3},... {n-1, n}

Example 19. Draw the graphs (a) C_3 (b) C_4 (c) C_5 (a) C_6, (e) C_8

Solution :



Example 20. How many vertices and how many edges do these graphs have (a) C_n (b) C_8 (c) Also find the degree sequence of C_4.

Solution :

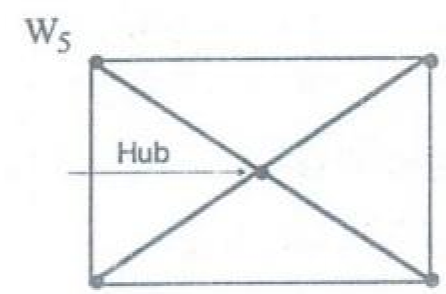
(a) n vertices, n edges

(b) 8 vertices, 8 edges.

(c) 2, 2, 2, 2

Definition: Wheel Graph :

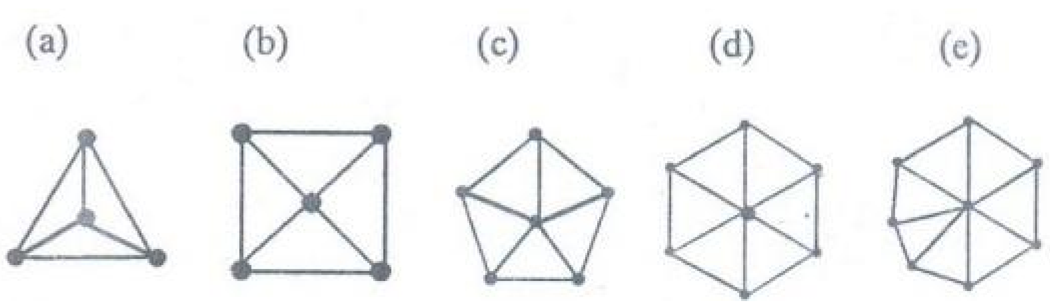
A wheel graph of order n is obtained by joining a new vertex called 'Hub' to each vertex of a cycle graph of order n 1, denoted by W_n.



Note: We obtain the wheel W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

Example 21. Draw the graphs (a) W_3, (b) W_4 (c) W_5, (d) W_6, (e) W_7

Solution :



Example 22. How many vertices and how many edges do these graph have (a) W_n (b) W_5 also find the degree sequence of W_4

Solution :

(a) n + 1 edges, 2n edges

(b) 5+1 = 6 vertices, (2)(5) = 10 edges

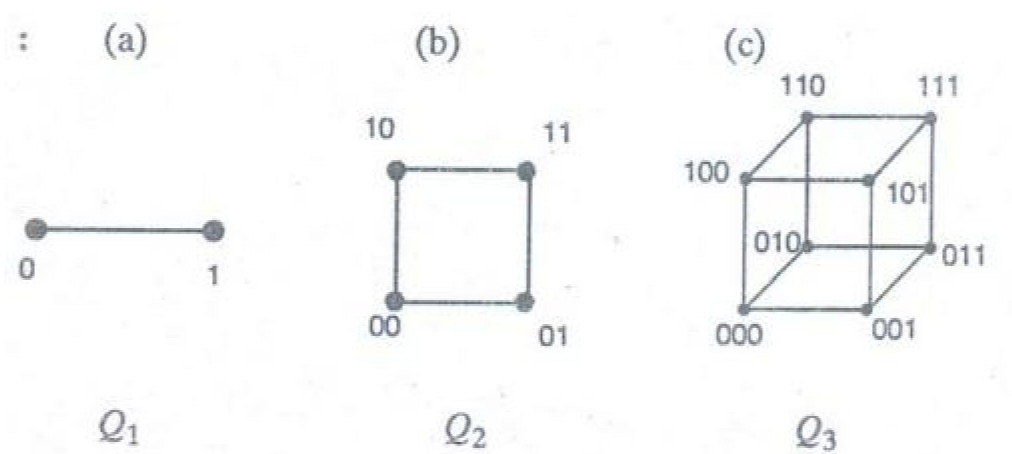
(c) 4, 3, 3, 3, 3

Definition: n-Cubes :

The n-dimensional hypercube, or n-cube, denoted by Q_n; is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

Example 23. Draw the graphs (a) Q_1 (b) Q_2 (c) Q_3

Solution :



Example 24. How many vertices and how many edges do those graphs here?

(a) Q_n (b) Q_3

Solution :

(a) 2^n vertices, $n\,2^{n-1}$ edges

(b) $2^3 = 8$ vertices, (3) $(2^2) = 12$ edges

Example 25. Find the degree sequence of Q_3.

Solution: 3, 3, 3, 3, 3, 3, 3

Regular graph: [A.U N/D 2012]

A graph in which all vertices are of equal degree is called a regular graph.

If the degree of each vertex is r, then the graph is called a regular graph of degree r.

Note :

(1) Every null graph is regular of degree zero.

(2) The complete graph k_n of degree n - 1.

(3) If G has n vertices and is regular of degree r, then G has (½) r n edges.

Example 26. What is the size of an r-regular (p, q) graph?

Solution : By the definition of regularity of G,

we have $\deg_G(v_1) = r$ for all $v_1 \epsilon V(G)$

But $2q = \Sigma \deg_G(v_1)$

$= \Sigma r$

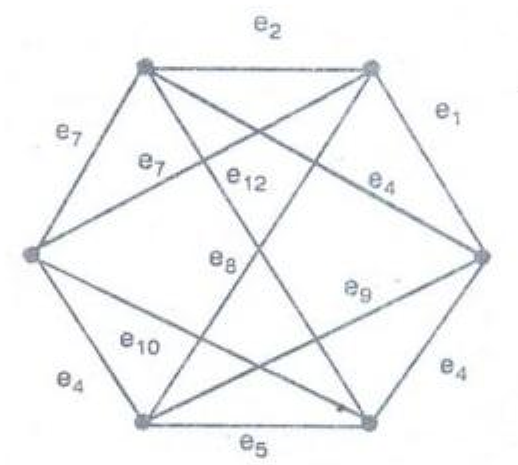
$= pr$

$q = pr/2$

Example 27. Does there exists a 4-regular graph on 6 vertices if so construct a graph.

Solution: We know that $q = pr/2 = (6)(4)/2 = 12$

Four regular graph on 6 vertices is possible and it contains 12 edges.



Example 28. For which values of n are these graphs regular? (a) K_n (b) C_n (c) W_n (d) Q_n

Solution:

(a) For all $n \geq 1$

(b) For all $n \geq 3$

(c) For $n = 3$

(d) For all $n \geq 0$

Example 29. How many vertices does a regular graph of degree four with 10 edges have?

Solution: Given

$r = 4$

$q = 10$

To find : p.

We know that $2q = pr$

$P = 2q/r$

$P = (2)(10)/4$

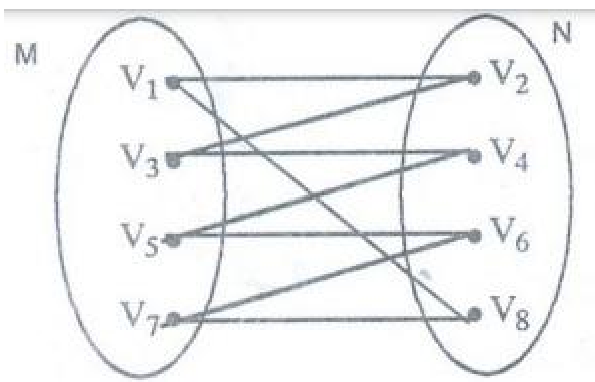
$p = 5$

Definition Bipartite graph

A bipartite graph is an undirected graph whose set vertices can be partitioned into two sets M and N is such a way that each edge joins a vertex in M to a vertex in N and no edge joins either two vertices in M or two vertices in N.

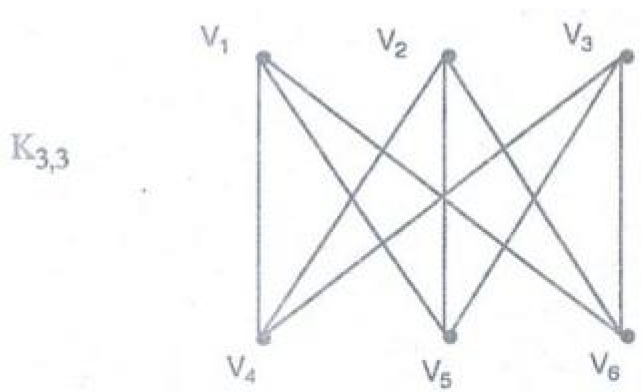
here V = MUN ; $M \cap N = \phi$

M = {V_1, V_3, V_5, V_7} ; V = {V_2, V_4, V_6, V_8}



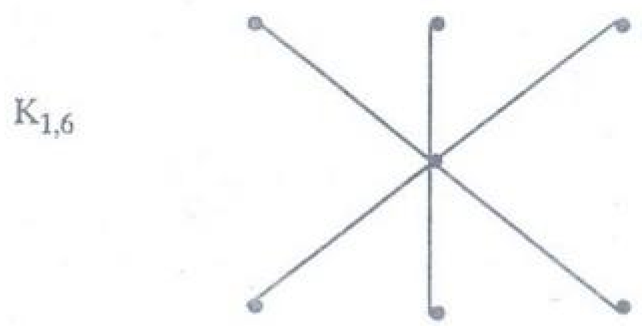
Definition: Complete Bipartite graph

A complete bipartite graph is a bipartite graph in which every vertex of M is adjacent to every vertex of N. The complete bipartite graphs that may be partitioned into sets M and N as above s.t M = m and |N| = n are denoted by $K_{m, n}$



Definition: Star graph

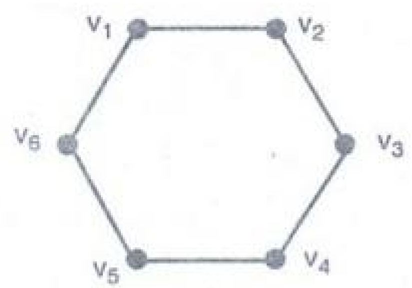
Any graph that is $K_{1, n}$ is called a star graph.



Example 30. Show that C_6 is a bipartite graph ?

Solution: The vertex set of C_6 can be partioned into the two sets

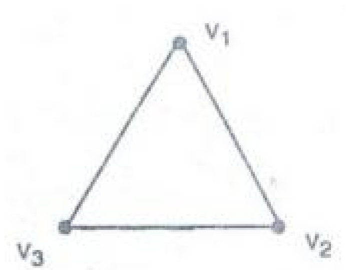
$V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and every edge of C_6 connects a vertex in V_1 and a vertex in V_2



Hence C_6 is a bipartite graph.

Example 31. Is K_3 is bipartite?

Solution: No, the complete graph K_3 is not bipartite.



If we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices.

If the graph is bipartite, these two vertices should not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.

K_3 is not bipartite.

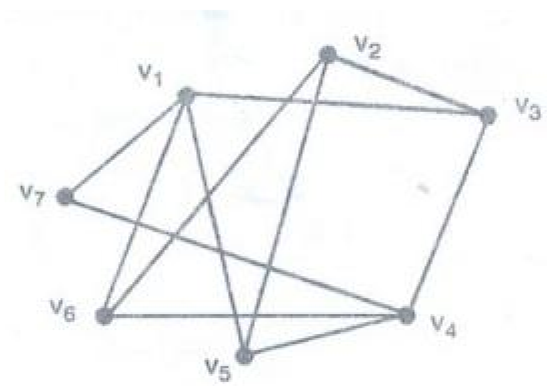
Example 32. How many vertices and how many edges of $K_{m,n}$ graph have?

Solution: m + n vertices, mn edegs.

Example 33. Find the degree sequence of the following graph $K_{2,3}$

Solution: 3, 3, 2, 2, 2

Example 34. Show that the following graph G is bipartite.



Solution: Graph G is bipartite since its vertex set is the union of two disjoint sets, $\{v_1, v_2, v_3\}$ and $\{v_3, v_5, v_6, v_7\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset.

Example 35. For which values of m and n is $K_{m,n}$ regular?

Solution: A complete bipartite graph $K_{m,n}$ is not a regular it $m \neq n$

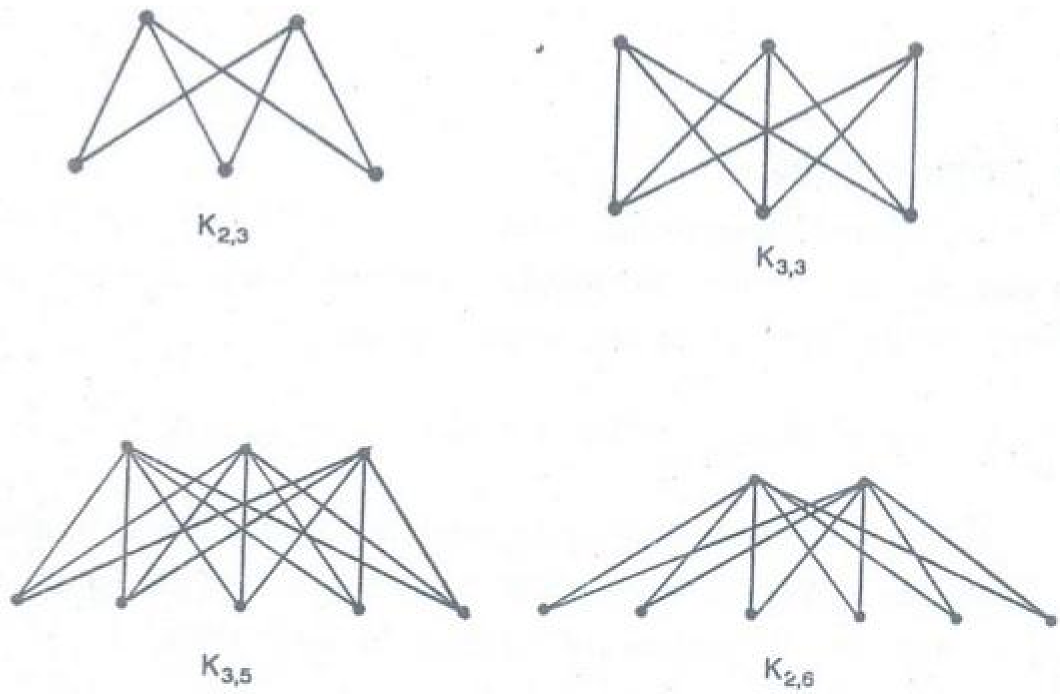
→ If m = n then $K_{m,n}$ is regular.

Example 36. Prove that a graph which contains a triangle can not be bipartite.

Proof: Atleast two of the three vertices must lie in one of the bipartite sets because there two are joined by edge, the graph can not be bipartite.

Example 37. Draw the complete bipartite graphs $K_{2,3}$, $K_{3,3}$ $K_{3,5}$ and $K_{2,6}$

Solution :



Example 38. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$

Solution: Let G be a complete bipartite graph with v vertices.

Let v_1 and v_2 be the number of vertices in the partitions V_1 and V_2 of vertex set of G.

Since G is complete bipartite, each vertex in V_1 is joined to each vertex in V_2 by exactly one edge.

Thus G has $v_1 v_2$ edges when $v_1 + v_2 = v$

But we know that maximum value of $v_1 v_2$ subject to

$v_1 + v_2 = v$ is $v^2/4$

Thus the maximum number of edges in G is $v^2/4$

ie. $e \leq v^2/4$

Graph coloring

The assign of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned different colors is called the proper coloring of G or simply vertex coloring.

If G has n coloring, then G is said to be n-colorable.

Theorem: A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof: Let G = (V,E) is a bipartite simple graph. Then $V = V_1 U V_2$, where V_1 and V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2.

If we assign one color to each vertex in V_1 and a second color to each vertex in V_2, then no two adjacent vertices are assigned the same color.

Suppose that it is possible to assign colors to the vertices of the graph using just wo colors.

→ No two adjacent vertices are assigned the same color.

Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other colour. Then, V_1 and V_2 are disjoint and $V = V_1 U V_2$.

i.e., every edge connects a vertex in V_1 and a vertex in v_2 since no two adjacent vertices are either both in V_1 or both in V_2. Consequently, G is bipartite.

Job Assignment problem

A company receives 5 applications to fill 6 vacant positions. Applicant x_1 is qualified to fill positions P_1 and P_4. Applicant x_2 is qualified to fill positions P_1, P_2, P_3, P_4 and P_5.

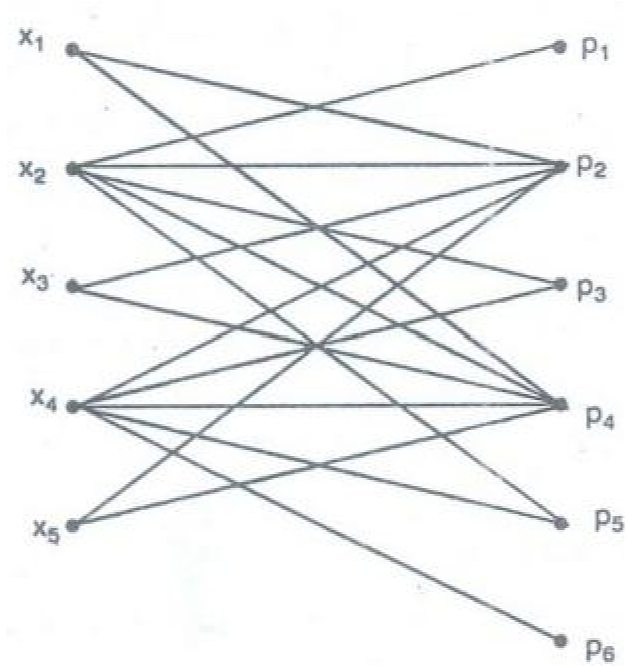
Applicant x_3 is qualified to fill positions P_2 and P_4. Applicant x_4 is qualified to fill positions. P_2, P_3, P_4, P_5 and P_6. Applicant x_5 is qualified to fill positions P_2 and P_4.

The problem is: Is it possible to recruit all the applicants and assign a job for which he is qualified?

We introduce a graph model for this with a vertex representing an applicant or a job and if an applicant is qualified for a job then we introduce an edge between the corresponding vertices.

Now, the problem becomes, finding a matching in this graph which is incident with all the x j s.

Such a matching does not exist in the graph, since the neighbour set of x_1, x_3 and x_5 is the same set {P_2, P_4}, that is, the applicants x_1, x_3 and x_5 are qualified only for 2 jobs, P_2 and P_4. Thus one of the 3 applicants cannot be matched.



Hence we cannot recruit all the five applicants and assign a job for which they are qualified.

Local Area Networks :

In various computers in a building, such as minicomputers and personal computers, as well as peripheral devices such as printers and plotters, can be connected using a local area network.

Some of these networks are based on a star topology, where all devices are connected to a central control device.

Parallel processing :

Computers made up of many separate processors, each with its own memory, helps overcome the limitations of computers with a single processor.

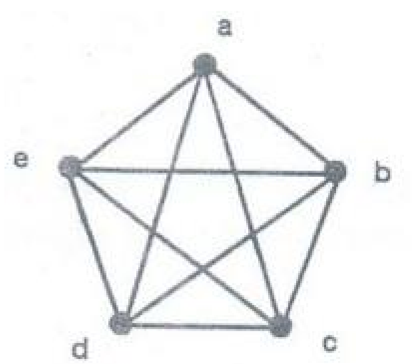
Parallel algorithms, which break a problem into a number of subproblems that can be solved concurrently, can then be devised to rapidly solve problems using a computer with multiple processors.

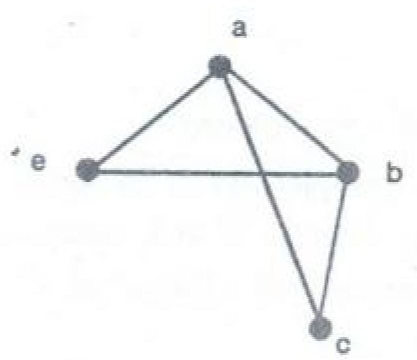
Definition: Subgraph

A subgraph of a graph $G = (V,E)$ is a graph $H = (W, F)$, where $W \leq V$ and $F \leq E$. A subgroup H of G is a proper subgraph of G if $H \neq G$.

Example 39. Draw two subgraph of K_5.

Solution:



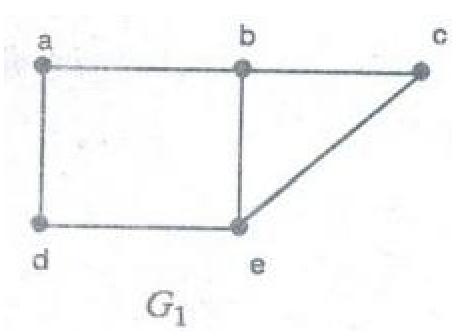


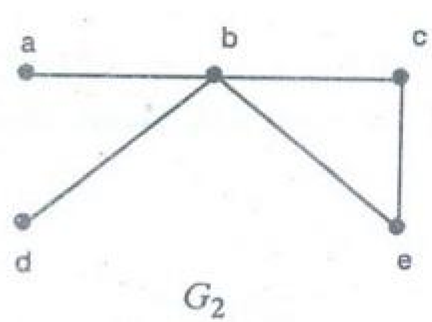
Example 40. How many subgraphs with atleast one vertex does K_3 have ?

Solution: 17

Definition: The union two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example 41. Find the union of the graphs G_1 and G_2

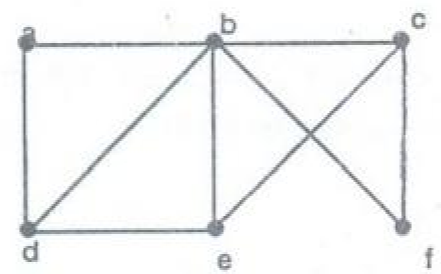




Solution : The vertex set of G_1 = {a, b, c, d, e}

The vertex set of G_2 = {a, b, c, d, f}

$G_1 U G_2$ = {a, b, c, d, e, f}



$G_1 \cup G_2$

Definition: Complement

The complement of G is defined as a simple graph with the same vertex set as G and value two vertices u and v are adjacent only when then are not adjacent in G.

Example 42. If the simple graph G has v vertices and e edges, how many edges does have?

Solution: have the edges.

$= v\,(v\text{-}1) / 2e$

Example 43. Show that if G is a simple graph with n vertices, then the union of G and is K_n.

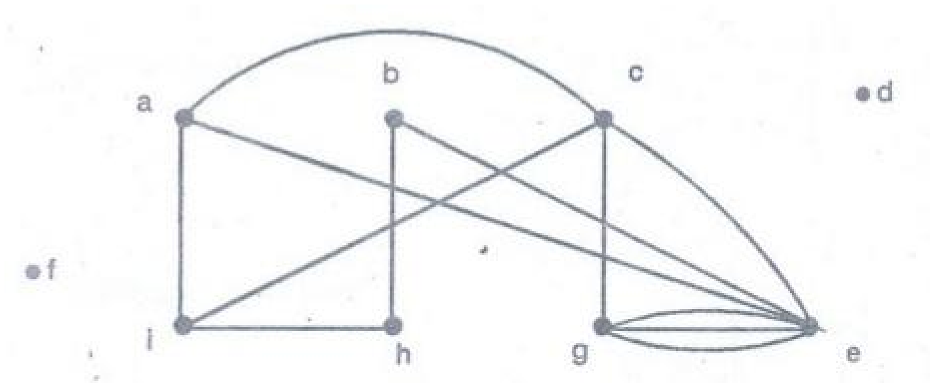
Solution: The union of G and contains an edge between each pair of the n vertices. Hence this union is K_n.

EXERCISE 3.2

1. Can a simple graph exist with 15 vertices each of degree five?

[Ans. No. because the sum of the degree of the vertices cannot be odd.]

2. Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.



[Ans. v = 9, e = 12; deg (a) = 3, deg (b) = 2, deg (c) = 4, deg (d) = 0, deg(e) 6, deg (f) = 0; deg (g) = 4; deg(h) = 2; deg(i) = 3, d and *f* are isolated.]

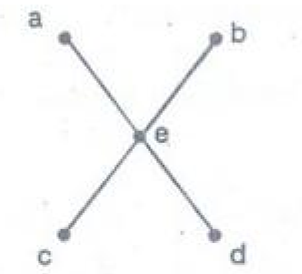
3. What does the degree of a vertex represent in a collaboration graph? What do isolated and pendant vertices represent?

[Ans. The number of colloaborators v has; someone who has never collaborated; someone who has just one collaborator.

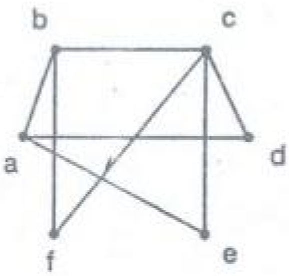
4. What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the word? What do isolated and pendant vertices in this graph represent ? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model ?

5. Determine whether the graph is bipartite.

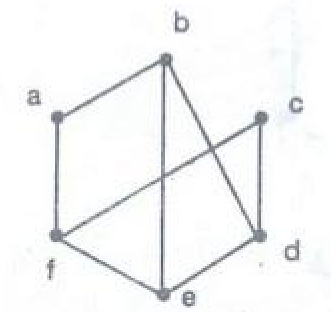
(a)



(b)



(c)

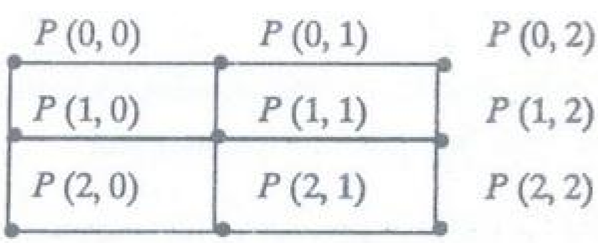


6. For which values of n are these graphs bipartite?

(a) K_n (b) C_n (c) W_n (d) Q_n

7. Draw the mesh network for interconnecting nine parallel processors.

[Ans.

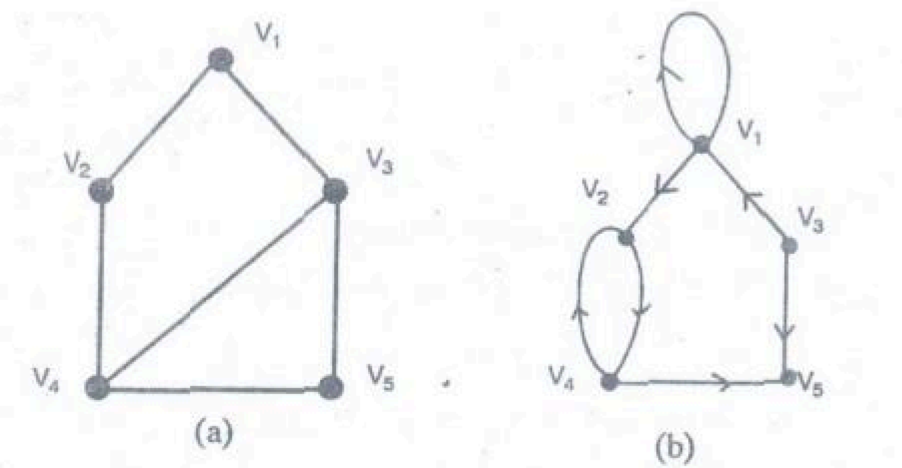


9. Show that every pair of processors in a mesh network of $n = m^2$ processors can communicate using $O(\sqrt{n}) = O(m)$ hops between directly connected processors.

Representing Graphs and Graph Isomorphism

Definition: Matrix representation of graphs and Digraphs:

We can represent a simple graph in the form of edge list or in the form of adjacency lists which are may be useful in computer programming.



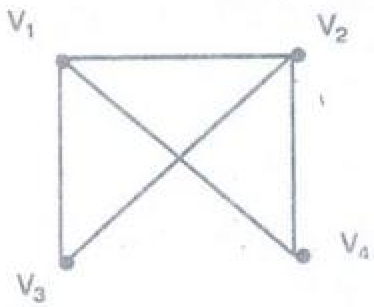
An Edge list of (a)		An Edge list of (b)	
Vertex	Adjacency vertices	Vertex	Adjacency vertices
V_1	V_2, V_3	V_1	V_1, V_2
V_2	V_1, V_4	V_2	V_4
V_3	V_1, V_4, V_5	V_3	V_1, V_5
V_4	V_2, V_3, V_5	V_4	V_2, V_5
V_5	V_3, V_4	V_5	–

Definition: Adjacency matrix

Let G (V, E) be a simple graph with n. Vertices ordered from V_1 to V_n, then the adjacency matrix A = $[aij]_{n \times n}$ of G is an n×nsymmetric matrix defined by the elements.

$$a_{ij} = \begin{cases} 1 & \text{when } V_i \text{ is adjacent to } V_j \\ 0 & \text{Otherwise} \end{cases}$$

It is denoted by A (G) or A_G



$$A_G = \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} \begin{pmatrix} V_1 & V_2 & V_3 & V_4 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Properties of adjacency matrix

1. An adjacency matrix completely defines a simple graph

2. The adjacency matrix is symmetric

3. Any element of the adjacency matrix is either 0 or 1, therefore it is also called as, bit matrix or boolean matrix.

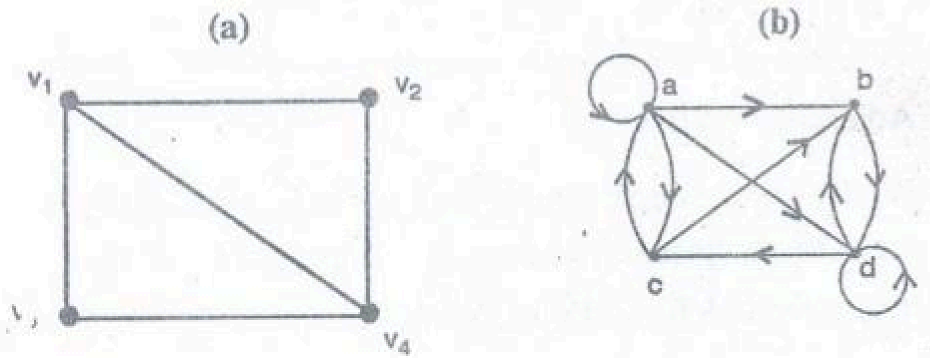
4. The i_{th} row in the adjacency matrix is determined by the edges which originate in the node V_i.

5. If the graph G is simple, the degree of the vertex V; equals the number of 1's in the i_{th} row (or i_{th} column) of A_G.

6. Given an n×n symmetric boolean matrix A, we can find a simple graph G s.t A is the adjacency matrix of G.

7. G is null ↔ A(G) is the zero matrix of order n.

Example 1. Use an adjacency list to represent the given graph.



Solution :

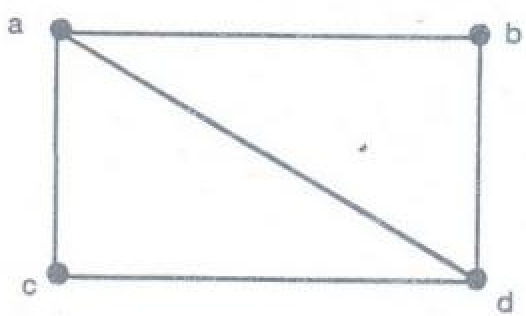
(a)

Vertex	Adjacent vertices
v_1	v_2, v_3, v_4
v_2	v_1, v_4
v_3	v_1, v_4
v_4	v_1, v_2, v_3

(b)

Vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

Example 2. Represent the following graph with an adjacency matrix.

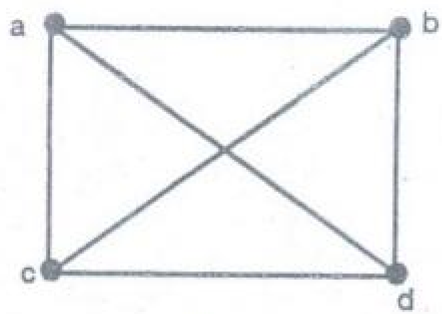


Solution :

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example 3. Write the adjacency matrix of K_4

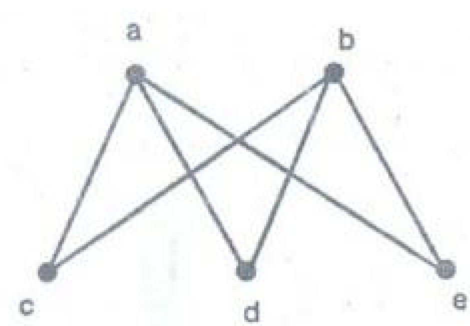
Solution: K_4 graph is



	a	b	c	d
a	0	1	1	1
b	1	0	1	1
c	1	1	0	1
d	1	1	1	0

Example 4. Write the adjacency matrix of $K_{2,3}$

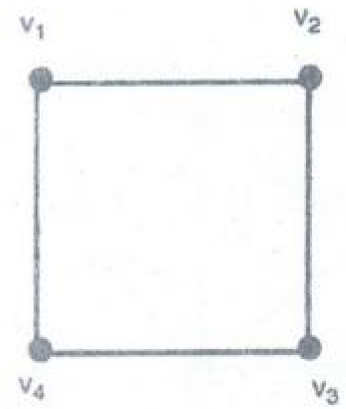
Solution: $K_{2,3}$ graph is



	a	b	c	d	e
a	0	0	1	1	1
b	0	0	1	1	1
c	1	1	0	0	0
d	1	1	0	0	0
e	1	1	0	0	0

Example 5. Write the adjacency matrix of C_4.

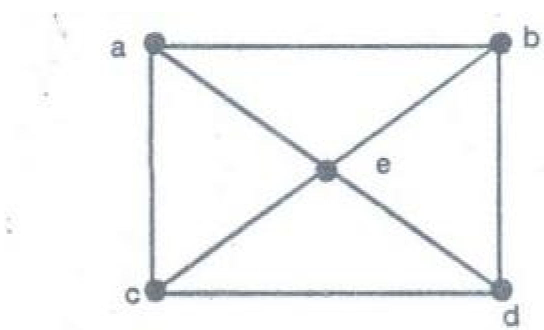
Solution : C_4 graph is



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Example 6. Write the adjaency matrix of W_4.

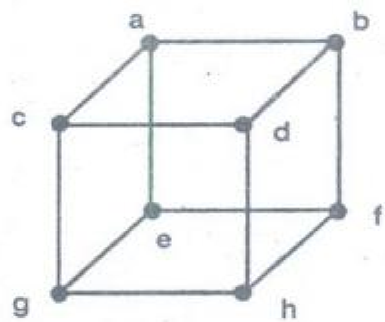
Solution: W_4 graph is



$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Example 7. Write the adjacency matrix of Q_3

Solution: The Q_3 graph is



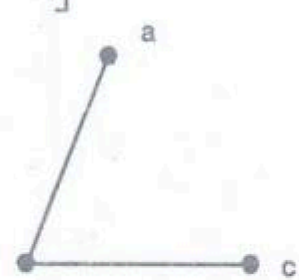
	a	b	c	d	e	f	g	h
a	0	1	1	0	1	0	0	0
b	1	0	0	1	0	1	0	0
c	1	0	0	1	0	0	1	0
d	0	1	1	0	0	0	0	1
e	1	0	0	0	0	1	1	0
f	0	1	0	0	1	0	0	1
g	0	0	1	0	1	0	0	1
h	0	0	0	1	0	1	1	0

Example 8. Draw a graph of the given adjacency matrix.

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

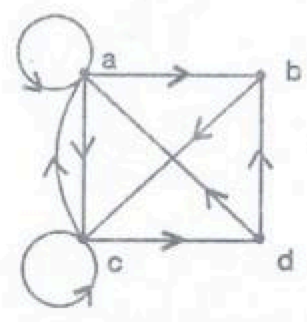
Solution : (a) Let

	a	b	c
a	0	1	0
b	1	0	1
c	0	1	0

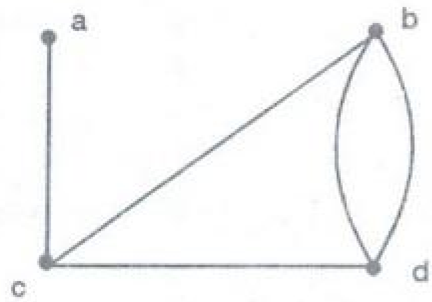


(b) Let

	a	b	c	d
a	1	1	1	0
b	0	0	1	0
c	1	0	1	0
d	1	1	1	0



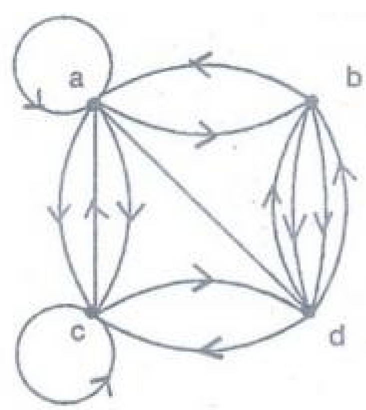
Example 9. Represent the given graph using an adjacency matrix.



Solution :

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

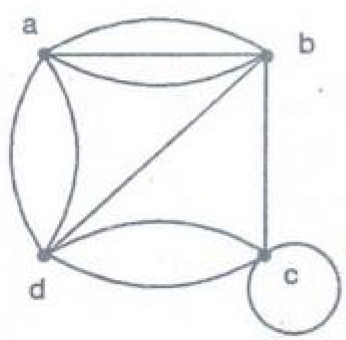
Example 10. Find the adjacency matrix of the given directed multigraph.



Solution :

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

Example 11. Use an adjacency matrix to represent the pseudograph shown in figure.



Solution: The adjacency matrix using the ordering of vertices a, b, c, d is

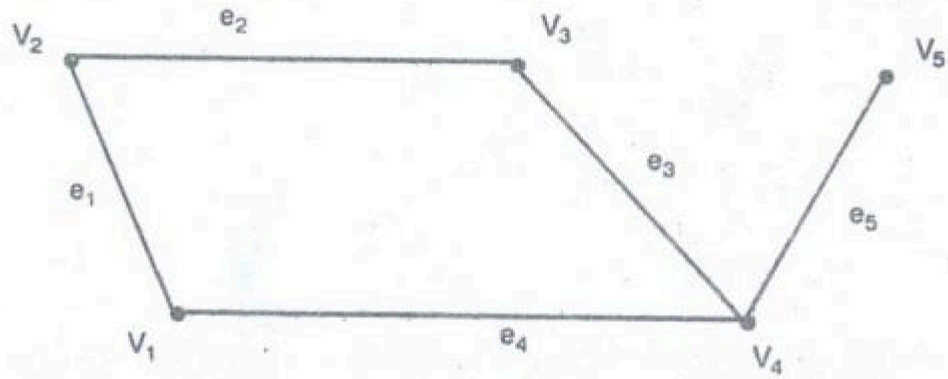
	a	b	c	d
a	0	3	0	2
b	3	0	1	1
c	0	1	1	2
d	2	1	2	0

Definition: Incidence matrix

Let G be a graph with n vertices, Let V = {V_1, V_2, ... V_n} and E = (e_1, e_2,...e_m). Define n x m matrix

I_G = $[mij]_{n \times m}$ where

$$m_{ij} = \begin{cases} 1 \text{ when } V_i \text{ is incident with } e_j \\ 0 \text{ Otherwise} \end{cases}$$



$$I_G = \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note :

1. The incidence matrix contains only 0 and 1.

2. The number of 1's in each row equals to the degree of the corresponding vertex.

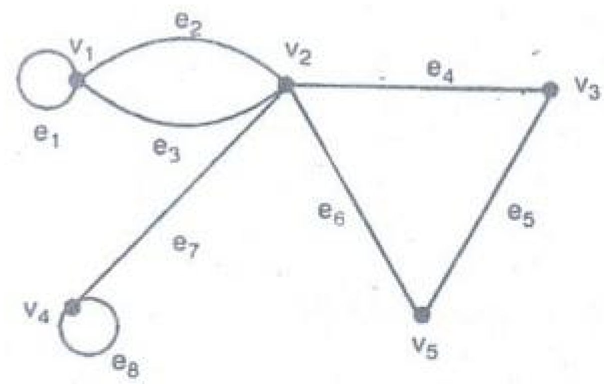
3. A row with all zeros represents an isolated vertex.

4. Every edge is incident on exactly two vertices, each column of the Incidence matrix has exactly two ones except the loop.

5. The parallel edges in a graph produce idential columns in its incidence matrix.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and the edges of the same graph.

Example 12. Represent pseudograph shown in figure using an incidence matrix.



Solution: The incidence matrix for this graph is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

Example 13. What is the sum of the entries in a row of the incidence matrix for an undirected graph?

Solution: Sum is 2 if e is not a loop, 1 if e is a loop.

Example 14. Find the incidence matrices for the graphs (a) K_n (b) C_n (c) W_n

Solution:

(a) $\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ & & & & 1 & 0 & \cdots & 0 \\ & & B & & 0 & 1 & \cdots & 0 \\ & & & & \vdots & \vdots & & \vdots \\ & & & & 0 & 0 & \cdots & 1 \end{bmatrix}$

Where B is the answer to (b)

Definition: Isomorphic Graphs

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function *f* from V_1 to V_2 with the property that a and b are adjacent in G1 if and only if *f(a)* and *f(b)* are adjacent in G_2, for all a and b in V_1. Such a function *f* is called an isomorphism.

In otherwords

Two graphs G and G' are isomorphic if there is a function *f*: V (G) → V (G') from the vertices of G to the vertices of G' such that

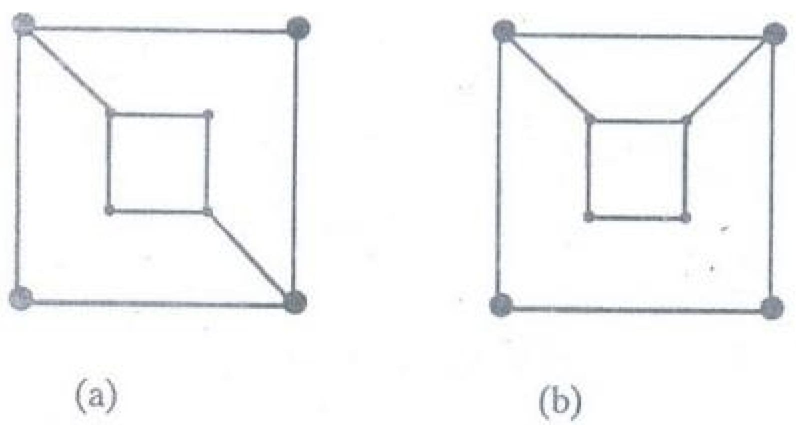
(i). f is one-to-one

(ii) f is onto and

(iii) For each pair of vertices u and v of G [u, v] ϵ E (G) ↔ [f(u), f(v)] ϵ E (G')

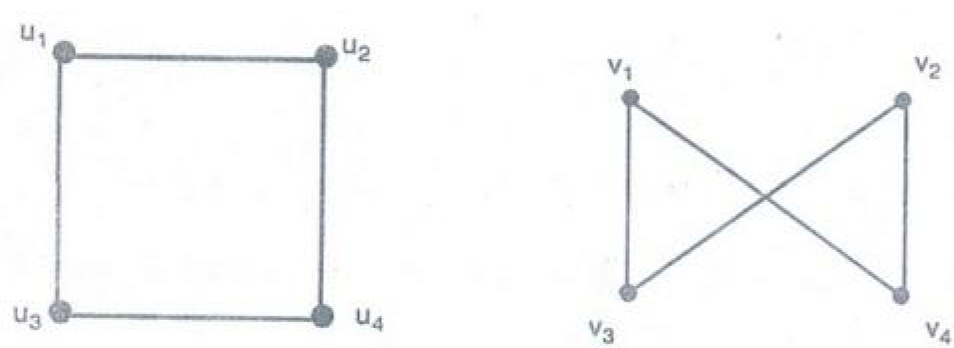
Any function f with the above three properties is called an isomorphism from G to G'.

Example 15. Explain why the two graphs given below are not isomorphic.



Solution: In the graph (a), no vertices of degree two are adjacent while in the graph (b) vertices of degree two are adjacent. Because isomorphism preserves adjacency of vertices, the graphs are not isomorphic.

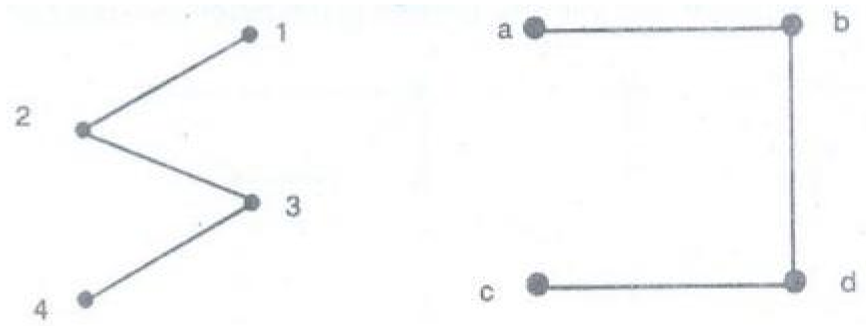
Example 16. Show that the graphs G = (V, E) and H = (W, F), shown in figure are isomorphic.



Solution: The function f with $f(u_1) = 1$, $f(u_2) = 4$, $f(u_3) = 3$, and $f(u_4) = 2$ is a one-to-one corresponding between V and W.

Here the correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4 and each of the pairs $f(u_1) = 1$ and $f(u_2) = 4$. $f(u_1) = 1$ and $f(u_3) = 3$, $f(u_2) = 4$ and $f(u_4) = 2$, and $f(u_3) = 3$ and $f(u_4) = 2$ are adjacent in H.

Example 17. Show that the two graphs shown in figure are isomorphic.



Solution: Here, $V(G_1) = \{1, 2, 3, 4\}$, $V(G_2) = \{a, b, c, d\}$,

$E(G_1) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ and $E(G_2) = \{\{a, b\}\}, \{b, d\}, \{d, c\}$

Define f: $V(G_1) \rightarrow V(G_2)$ as

$f(1) = a, f(2) = b, f(3) = d$ and $f(4) = c$

f is clearly one-one and onto, hence an isomorphism.

$\{1, 2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2, 3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3, 4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

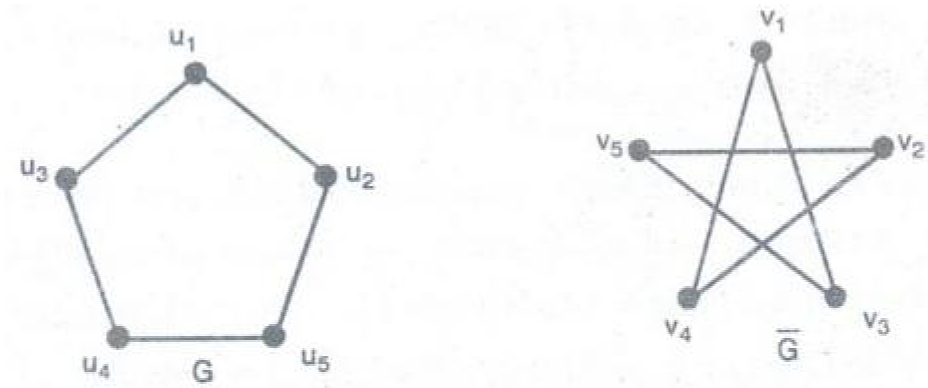
$\{1, 3\}$ € $E(G1)$ and $\{f(1), f(3)\} = \{a, d\}$ € $E(G_2)$

$\{1, 4\}$ € $E(G_1)$ and $\{f(1), f(4)\} = \{a, c\}$ € $E(G_2)$

$\{2, 4\}$ € $E(G_1)$ and $\{f(2), f(4)\} = \{b, c\}$ € $E(G_2)$

Hence f preserves adjacency as well as non-adjacency of the vertices. Therefore G_1 and G_2 are isomorphic.

Example 18. Prove that the graphs G and G given below are isomorphic



Solution: The two graphs have the same number of vertices same number of edges and same degree sequence consider the function *f*.

$f(u_1)$ V_1, $f(u_2) = v_3$, $f(u_3) = V_4$, $f(u_4) = V_2$, $f(u_5) = v_5$

then the adjacency matrices of the two graphs corresponding to *f* are

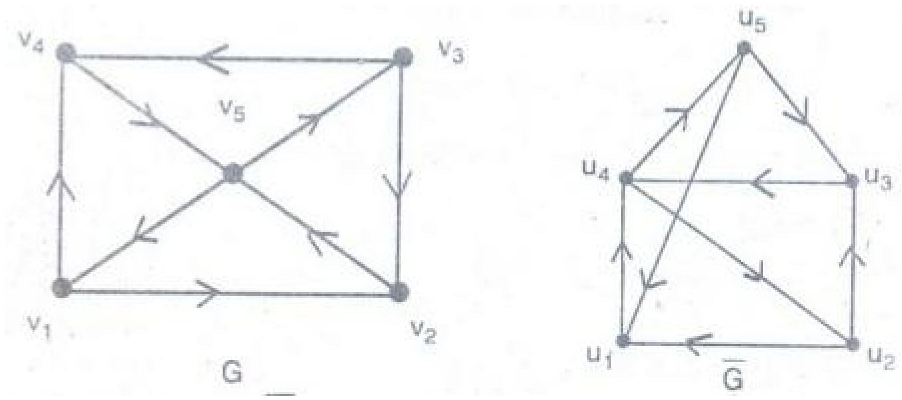
$$A(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$A(\overline{G}) = \begin{array}{c} \\ v_1 \\ v_3 \\ v_4 \\ v_2 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_3 & v_4 & v_2 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$A(G) = A(\overline{G})$$

G and Ğ are isomorphic to each other.

Example 19. Show that the Digraphs are isomorphic.



Solution: G and   are having 5 vertices and 8 edges. Consider indegree and out degree of the vertices if G and .

G	deg + in degree	deg – out degree
v_1	1	2
v_2	2	1
v_3	1	2
v_4	2	1
v_5	2	2

$\bar{G}$	deg+ in degre	deg – out degree
u_1	2	1
u_2	1	2
u_3	2	1
u_4	2	2
u_5	1	2

Now

$f(v_1) = u_5, f(v_2) = u_1, f(v_3) = u_2$

$f(v_4)=u_3, f(v_5) = u_4$

Clearly f is one to one and onto.

$\Rightarrow A_G = A_{\breve{G}}$ under this mapping f

G and Ğ are isomorphic.

Example 20. Prove that any 2 simple connected graphs with n vertices all of degree 2 are isomorphic.

Solution: We know that total degree of a graph is given by

$\Sigma_{ni=1}$ $d(V_i)=2|E|$

then $|V|$ = number of vertices n

$|E|$ = number of edges

Further the degree of every vertex is 2.

Therefore $\Sigma_{ni=1}$ $2 = 2|E|$

$2 ((n) - 1 + 1) = 2 |E|$

$\Rightarrow n = |E|$

number of edges = number of vertices. Therefore the graphs are cycle graphs Hence they are isomorphic.

Example 21. Can a simple graph with 7 vertices be isomorphic to its complement?

Solution: A graph with 7 vertices can have a maximum number of edges.

$= 7(7-1) / 2 = 7\times6 / 2 = 21 = 21$ edges

21 edges cannot be split into 2 equal integers. Therefore, G and cannot equal number of edges. Hence a graph with 7 vertices cannot be isomorphic to its complement.

Example 22. Let G be a simple graph all of whose vertices have degree 3 and $|E| = 2|V| - 3$. What can be said about G?

Solution :

$\Sigma_{|v|i=1}$ $d(V_i)=2|E|$

$3 (|V| - 1+1) = 2|E|$

$3 |V| = 2 |E|$

$\Rightarrow 3|V| = 2(2 |V| - 3)$

= 3|V| = 4|V| - 6

|V| = 6

Number of vertices in G = 6 It can be concluded that G is isomorphic to $K_{3,3}$

Example 23. Show that isomorphism of simple graphs is an equivalence relation.

Solution:

(i) Reflexive: G is isomorphic to itself by the identity function, so isomorphism is reflexive.

(ii) Symmetric: Suppose that G is isomorphic to H. Then there exists a one-to-one correspondence f from G to H that preserves adjacency and nonadjacency. From this f^{-1} is a one-to-one correspondence from H to G that preserve adjacency and non adjacency.

Hence, isomorphism is symmetric.

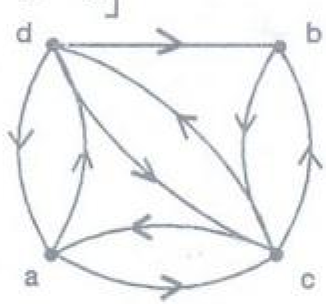
(iii) Transitive: If G is isomorphic to H and H is isomorphic to K, then there are one-to-one correspondences f and g from G to H and from H to K that preserve adjacency and non adjacency. It follows that gof is a one-to-one correspondence from G to K that preserves adjacency and non adjdacency. Hence, isomorphism is transitive.

EXERCISES 3.3

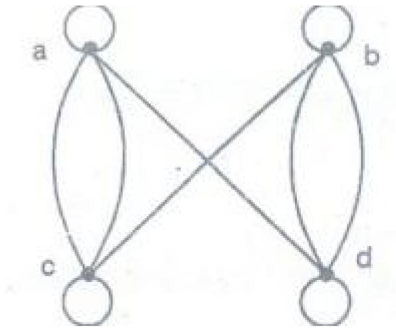
1. Draw a graph with the given adjacency matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

[Ans.



2. Represent the given graph using an adjacency matrix.

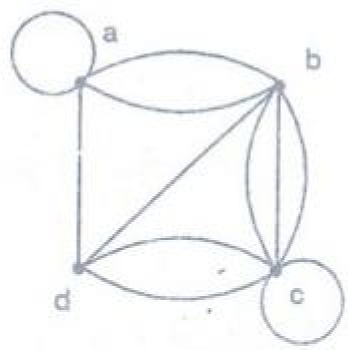


[Ans. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

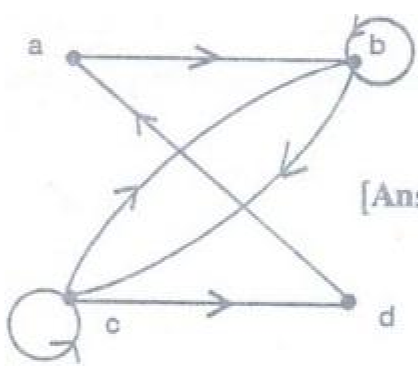
3. Draw an undirected graph represented by the given adjacency matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

[Ans.



4. Find the adjacency matrix of the given directed multigraph.

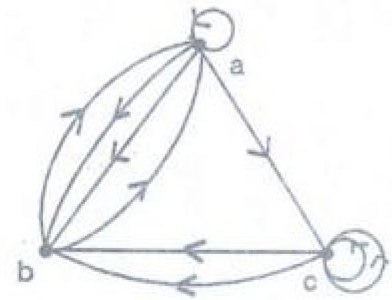


[Ans. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

5. Draw the graph represented by the given adjacency matrix.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

[Ans.



6. Is every zero-one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?

[Ans. Yes]

7. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertiex.

[Ans. Zeros]

8. Show that the vertices of a bipartite graph with two or more vertices can be ordered so that its adjacency matrix has the form

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

9. Find a self-complementary simple graph with five vertices.

[Ans. C_5]

10. For which integers n is C_n self-complementary?

[Ans. for n = 5 only]

11. How many non isomorphic simple graphs are there with five vertices and three edges ?

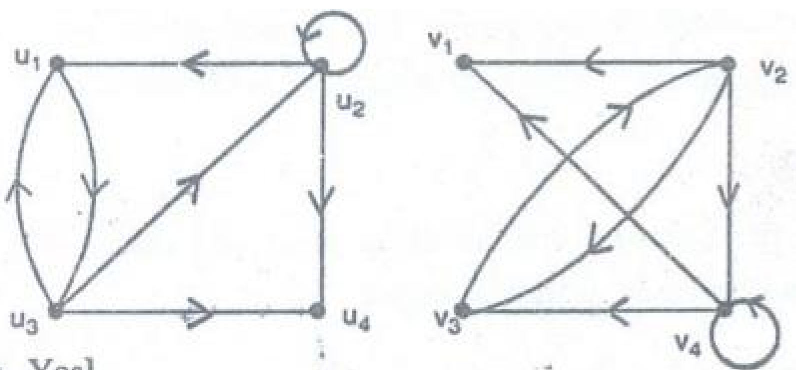
[Ans. 4]

12. Are the simple graphs with the following adjacency matrices isomorphic?

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ [Ans. Yes]

(b) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ [Ans. No]

13. Determine whether the given pair of directed graphs are isomorphic.



[Ans. Yes]

14. Find a pair of non isomorphic graphs with the same degree sequence such that one graph is bipartite, but the other graph is not bipartite.

[Ans. Many answers are possible for example C_6 and $C_3 \cup C_3$]

15. What is the product of the incidence matrix and its transpose for an undirected graph ?

[Ans. The product is $[a_{ij}]$ where a_{ij} is the number of edges from v_i to v_j when $i \neq j$ and a_{ii} is the number of edges incident to v_1]

CONNECTIVITY

Definition: Path

A path in a multigraph G consists of an alternating sequence of vertices and edges of the form

$V_0, e_1\ V_1, e_2, V_2, \ldots\ e_{n-1}, V_{n-1}, e_n, V_n$

where each edge e_i contains the vertices V_{i-1} and V_i

The number n of edges is called the length of the path.

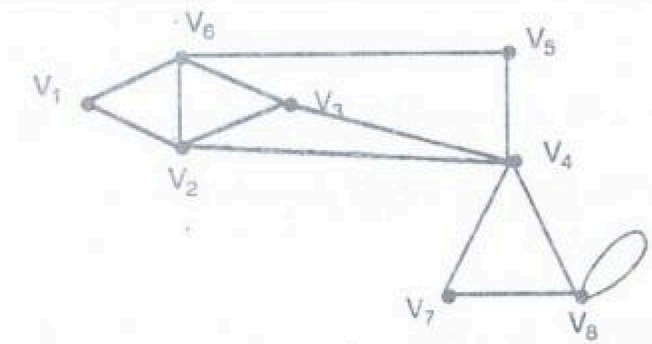
The path is said to be closed if $V_o = V_n$ we say the path is from V_o to V_n or between V_o and V_n or Connects V_o to V_n.

Note :

1. A simple path is a path in which all vertices are distinct. (A path in which all edges are district will be called a trail)

2. If $V_o = V_n$, then P is called a closed path. On the other hand if $V_o \neq V_n$, then P is an open path.

3. For the following graph



Path	Length	Simple path	Closed path	Circuit	Cycle
$V_1 - V_2 - V_6 - V_1$	3	Yes	Yes	Yes	Yes
$V_6 - V_2 - V_3 - V_6$	3	Yes	Yes	Yes	Yes
V_1	0	Yes	Yes	No	No
$V_8 - V_8$	1	Yes	Yes	Yes	Yes
$V_1 - V_2 - V_1$	2	No	Yes	No	No
$V_5 - V_4 - V_7 - V_8 - V_4 - V_3$	5	No	No	No	No

Definition: Circuit:

A path of length ≥ 1 with no repeated edges and whose end vertices are same is called a circuit.

Note :

4. A cycle is a circuit with no other repeated vertices except its end vertices.

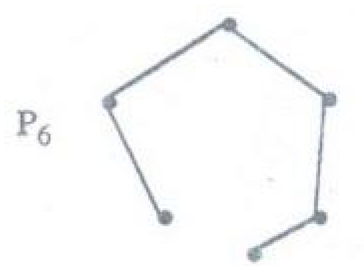
5. A cycle is a simple circuit, a loop is a cycle of length 1.

6. In a graph a cycle that is not a loop must have length atleast 3, but there may be cycles of length 2 in a multigraph.

7. Two paths in a graph are said to be edge-disjoint if they have no common edges, they are vertex - disjoint if they have no common vertices.

Definition: Path graph

A path graph of order 'n' is obtained by removing on edge from a C_n graph, denoted by P_n.



Definition: Trail

A trail from v to w is a path from v to w that does not contain a repeated edge.

Thus a trial from v to w is a path of the form

$v = v_0, e_1, v_1, e_2, \ldots v_{k-1}, e_k, v_k = w$ where all the e_i are distinct.

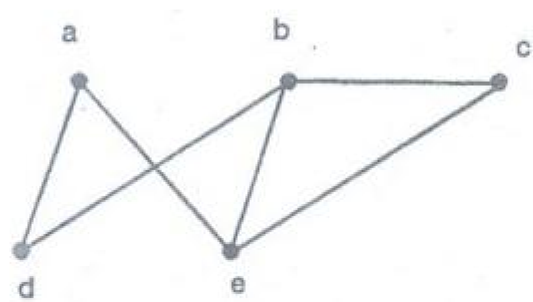
Note: Every simple path is also a trail, and every trail is also a path, but these inclusion do not reverse.

	Repeated edge	Repeated vertex	Starts and Ends at same point?
Path	allowed	allowed	allowed
Trail	no	allowed	allowed
Simple path	no	no	no
Closed path	allowed	allowed	yes
Cycle	no	allowed	yes
Simple cycle	no	first and last only	yes

Example 1: Does each of these lists of vertices from a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

(a) a, e, b, c, b (b) e, b, a, d, b, e

(c) a, e, a, d, b, c, a (d) c, b, d, a, e, c



Solution :

(a) path of length 4, not a circuit, not simple.

(b) not a path

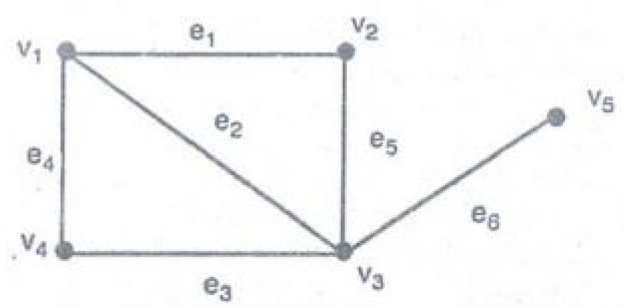
(c) not a path

(d) simple circuit of length 5.

Definition: Connected and disconnected graphs

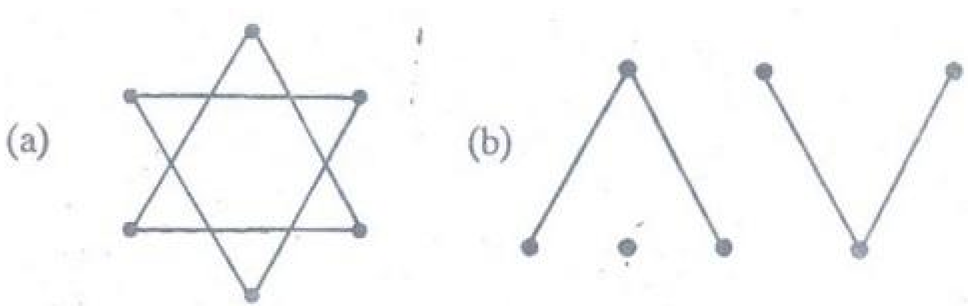
A graph G is a connected graph is there is atleast one path between every pair of vertices in G. Otherwise G is a disconnected graph.

Example:



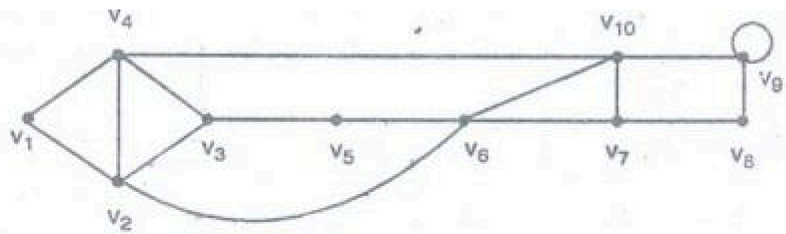
Note: A disconnected graph consists of two or more connected graphs. Eachof these connected subgroups is called a component (or a block).

Example 2. Determine whether the given graph is connected.



Solution: (a) No (b) No

Example 3. For the following graph



we have

Path	Length	Simple path	Closed path	Circuit	Cycle
$v_1-v_4-v_3-v_5-v_6-v_{10}-v_4-v_1$	7	no	yes	no	no
$v_2-v_3-v_5-v_6-v_7-v_{10}-v_6-v_2$	7	no	yes	yes	no
$v_1-v_2-v_1$	2	no	yes	no	no
$v_1-v_4-v_3-v_2-v_1$	4	yes	yes	yes	yes
v_9-v_9	1	yes	yes	yes	yes
v_1	0	yes	yes	no	no
$v_5-v_6-v_7-v_{10}-v_6-v_2$	5	no	no	no	no
$v_4-v_2-v_3-v_4$	3	yes	yes	yes	yes

Theorem: If a graph G (either connected or not) has exactly two vertices of odd degree, there is a path joining these two vertices.

Proof: Case (i) Let G be connected.

Let v_1 and v_2 be the only vertices of G with are of odd degree.

But we know that number of odd vertices is even.

Clearly there is a path connecting v_1 and v_2, because G is connected.

Case (ii) Let G be disconnected.

Then the components of G are connected.

Hence v_1 and v_2 should belong to the same component of G.

Hence, there is a path between v_1 and v_2.

Theorem :

The maximum number of edges in a simple disconnected graph G with n vertices and k components is (n - k) (n - k + 1)

Proof :

Let the number of vertices in the i_{th} component of G be n_1 ($n_i \geq 1$)

Then $$n_1 + n_2 + \dots + n_k = n \text{ or } \sum_{i=1}^{k} n_i = n$$

Hence, $$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$$\therefore \left\{\sum_{i=1}^{k} (n_i - 1)\right\}^2 = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i=!j} (n_i - 1)(n_j - 1) = n^2 - 2nk + k^2 \quad \dots (3)$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 - 2nk + k^2 \quad [\because (3) \geq 0, \text{ as each } n_i \geq 1]$$

$$\Rightarrow \sum_{i=!1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k \quad \dots (4)$$

Now the maximum number of edges in the i_{th} component of

G = ½ n_i (n_i − 1)

Therefore maximum number of edges of G

$$= \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n, \text{ by } (1)$$

$$\leq \frac{1}{2} (n^2 - 2n\,k + k^2 + 2n - k) - \frac{1}{2} n, \text{ by } (4)$$

$$\Rightarrow \leq \frac{1}{2} (n^2 - 2n\,k + k^2 + n - k)$$

$$\Rightarrow \leq \frac{1}{2} [(n - k)^2 + (n - k)]$$

$$\Rightarrow \leq \frac{1}{2} (n - k)(n - k + 1)$$

Example 3. Let G = (V,E) be a simple graph. Let R be the relation on V consisting of pairs of vertices (u, v) such that there is a path from u to v or such that u = v. Show that R is an equivalence relation.

Solution :

(i) Reflextive :

R is reflexive by definition.

(ii) Symmetry: Assume that (u, v) ϵ R, then there is a path from u to v. Then (v, u) ϵ R because there is a path from v to u.

(iii) Transitive:

Assume that (u, v) ϵ R and (v, w) ϵ R; then there are paths from u to v and from v to w. Putting these two paths together gives a path from u to w.

Hence, (u, w) ϵ R. It follows that R is transitive.

Theorem: There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof: Let u and v be two distinct vertices of the connected undirected graph G = (V, E).

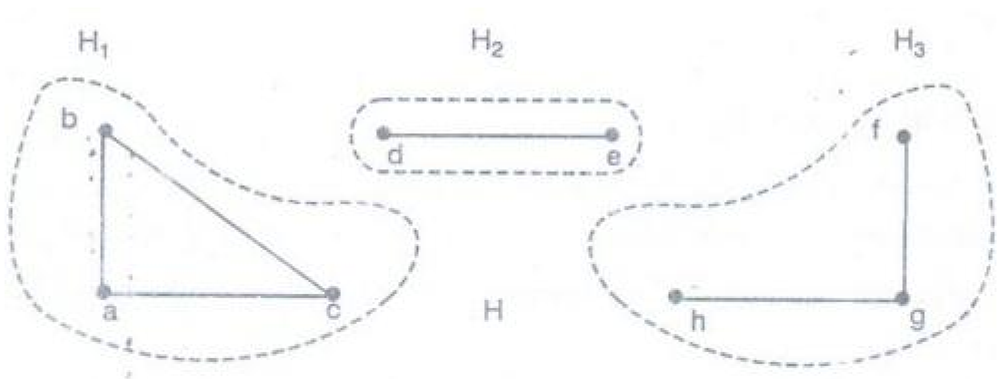
Since G is connected, there is at least one path between u and v. Let $x_0, x_1, \ldots, x_n$, where $x_0 = u$ and $x_n = v$, be the vertex sequence of a path of least length.

This path of least length is simple.

Suppose it is not simple. Then $x_i = x_j$ for some i and j with $0 \leq i < j$.

This means that there is a path from u to v of shorter length with vertex sequence $x_0, x_1, \ldots, x_{i-1}, x_j, \ldots, x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i, \ldots, x_{i-1}$.

Example 4. What are the connected components of the graph H shown in figure.



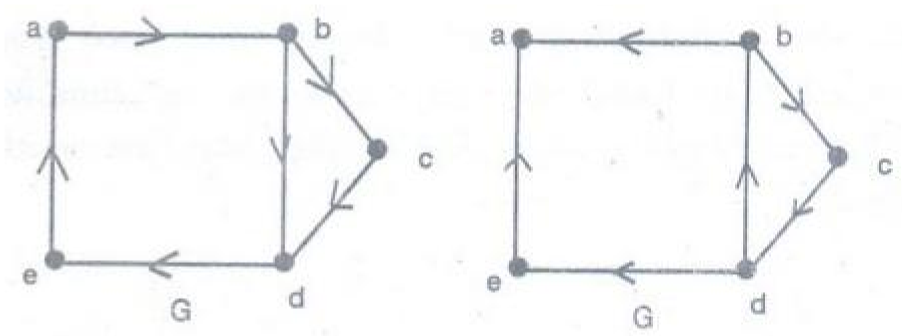
Solution: The graph H is the union of three disjoint connected subgraphs H_1, H_2, and H_3.

These three subgraphs are the connected components of H.

Definition: A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

Definition: A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Example 5. Are the directed graphs G and H shown in figure strongly connected? Are they weakly connected?



Solution: G is strongly connected because there is a path between any two vertices in this directed graph. Hence, G is also weakly connected.

The graph H is not strongly connected. There is no directed path from *a* to *b* in this graph. However, H is weakly connected, because there is a path between any two vertices in the underlying undirected graph of H.

Cut vertex, Cut set and Brige

A cut vertex of a connected graph G is a vertex whose removal increases the number of components. Clearly if v is a cut vertex of a connected graph G, G - v is disconnected. A cut vertex is also called a cut point.

Bridge:

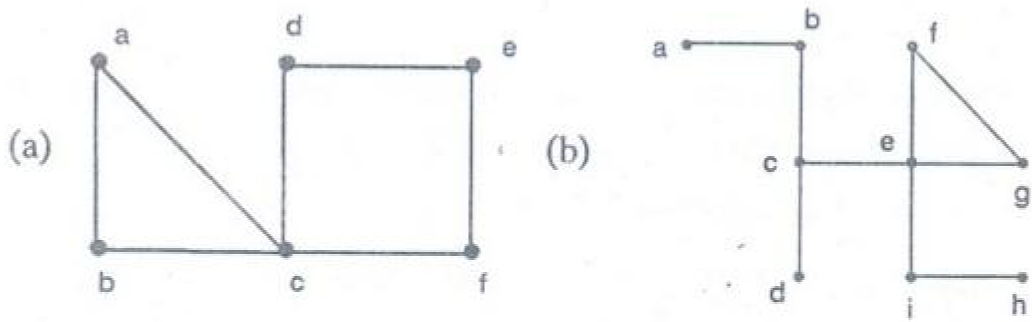
If a graph G is connected and e is an edge such that G - e is not connected, then *e* is said to be a bridge or a cut edge.

Paths in Acquaintanceship Graphs: In an acquaintanceship graph there is a path between two people if there is a chain of people linking these people, where two people adjacent in the chain know one another.

Paths in Collaboration Graphs: In a collaboration graph two vertices a and b, which represent authors, are connected by a path when there is a sequence of authors beginning at a and ending at b such that the two authors represented by the endpoints of each edge have written a joint paper.

Paths in the Hollywood graph: In the Holly wood graph two vertices a and b are linked when there is a chain of actors linking a and b, where every two actors adjacent in the chain have acted in the same movie.

Example 6. Find all the cut vertices of the given graph.



Solution :

(a) e

(b) b, c, e, i

Example 7. Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

Solution: If a vertex is pendant it is clearly not a cut vertex. A endpoint of a cut edge that is a cut vertex is not pendant.

Remove of a cut edge produces a graph with more connected components than in the original graph.

If an endpoint of a cut edge is not pendant, the connected component it is in after the remove cut edge contains more than just this vertex.

From this, removal of that vertex and all edges incident to it, including the original cut edge, produces a graph with more connected components than were in the original graph.

Hence, an endpoint of a cut edge that is not pendant is a cut vertex.

Theorem: Let G be a graph with adjacency matrix A with respect to the ordering 1, 2, ..., n (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length *r* from *i* to *j*, where *r* is a positive integer, equals the (i,j)th entry of A_r.

Example 8. Show that a simple graph G with n vertices is connected if it has more than (n - 1) (n - 2)/2 edges.

Solution:

Suppose that G is not connected.

Then it has a component of k vertices for some k, $1 \leq k \leq n-1$.

The most edges G could have is

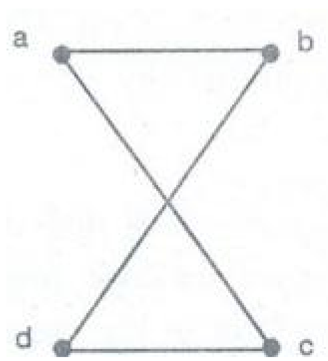
$C(k, 2) + C(n-k, 2) = [k(k-1) + (n-k)(n-k-1)]/2$

$= k^2 - nk + (n^2 - n)/2$.

This quadratic function off is minimized at $k = n/2$ and maximized at $k = 1$ or $k = n-1$.

Hence, if G is not connected, then the number of edges does not exceed the value of this function at 1 and at n-1, namely, $(n - 1)(n - 2)/2$.

Example 9. How many paths of length four are there from a to d in the simple graph G in figure.
[A.U N/D 2012]



Solution: The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Hence, the number of paths of length four from *a* to *d* is the (1, 4)th entry of A^4. Since

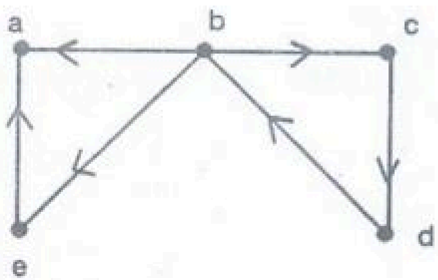
$$A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

there are exactly eight paths of length four from *a* to *d*. From this graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d ; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths from *a* to *d*.

EXERCISES 3.4

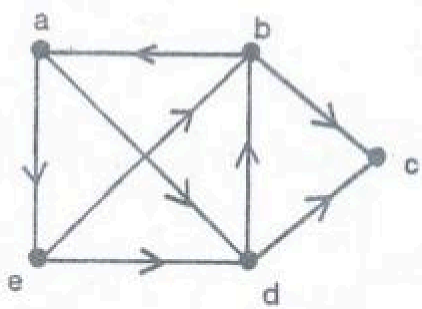
1. Determine whether each of these graphs is strongly connected and if not, whether it is weakly conected.

(a)



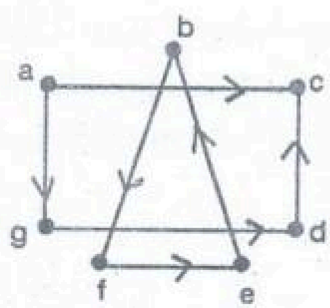
[Ans. Weakly connected]

(b)



[Ans. Weakly connected]

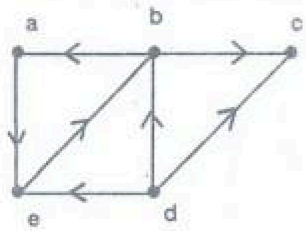
(c)



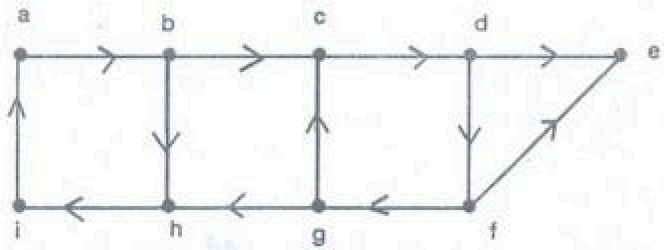
[Ans. Not strongly or Weakly connected]

2. Find the strongly connected components of each of these graphs.

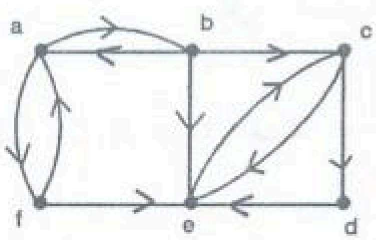
(a)



(b)

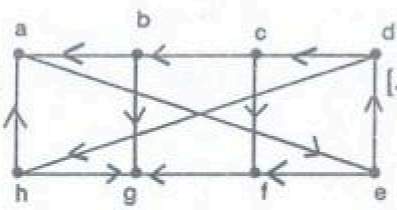


(c)



[**Ans.** $\{a, b, f\}$, $\{c, d, e\}$]

(d)



[**Ans.** $\{a, b, c, d, e, h\}$, $\{f\}$, $\{g\}$]

3. Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in this strongly connected component.

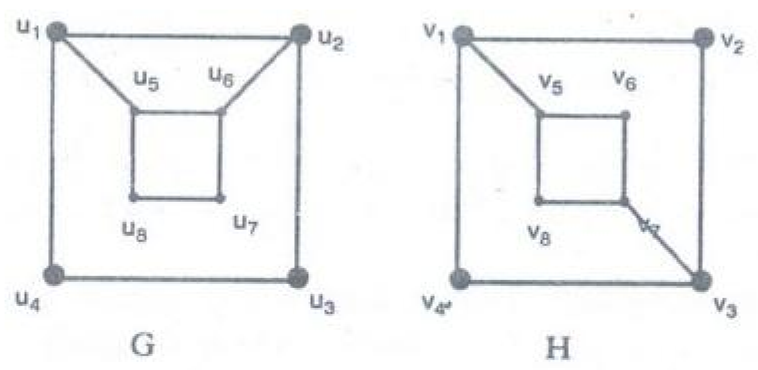
4. Find the number of paths of length n between two different vertices in K_4 if *n* is

(a) 2 [Ans. 2]

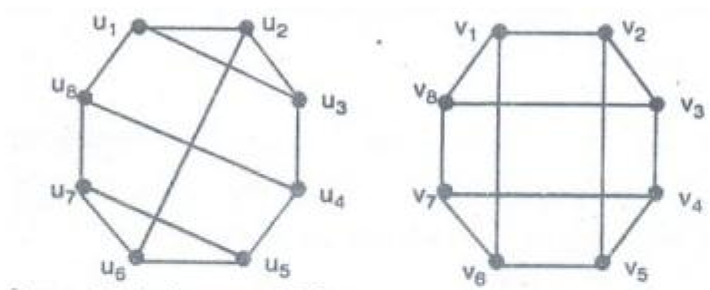
(b) 3 [Ans. 7]

(c) 4 [Ans. 20]

5. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

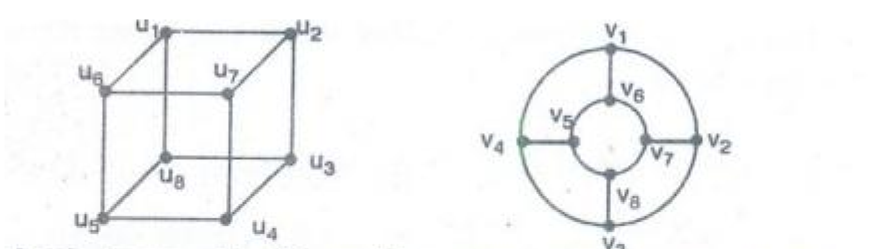


6. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



[Ans. Not isomorphic]

7. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



[Ans. Isomorphic (the path $u_1, u_2, u_7, u_6, u_5, u_4, u_3, u_8, u_1$ corresponds to the path $v_1, v_2, v_3, v_4, v_5, v_8, v_7, v_6, v_1$)

8. Show that every connected graph with n vertices has at least n-1 edges.

9. Show that in every simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

10. Show that a vertex *c* in the connected simple graph G is a cut vertex if and only if there are vertices *u* and *v*, both different from c, such that every path between *u* and *v* passes through *c*.

11. Show that a simple graph with at least two vertices has at least two vertices has at least two vertices that are not cut vertices.

12. Show that an edge in a simple graph is a cut edge if and only if this edge is not part of any simple circuit in the graph.

13. Show that if a connected simple graph G is the union of the graph G_1 and G_2, then G_1 and G_2 have at least one common vertex.

14. Show that if a simple graph G has *k* connected components and these components have $n_1, n_2, ..., n_k$ vertices, respectively, then the number of edges of G does not exceed

$\Sigma_{k i=1}$ C (n_i, 2)

15. How many nonisomorphic connected simple graphs are there with *n* vertices when *n* is

(a) 2?[Ans. 1] (b) 3? [Ans. 2] (c) 4? [Ans. 6] (d) 5? [Ans. 21]

16. Let P_1 and P_2 be two simple paths between the vertices *u* and in the simple graph G that do not contain the same set of edges. Show that there is a simple circuit in G.

17. Show that a simple graph G is bipartite if and only if it has no circuits with an odd number od edges.

18. Explain why in the collaboration graph of mathematicians a vertex representing a mathematician is in the same connected component as the vertex representing Paul Erdos if and only if that mathematician has a finite Erdos number.

19. What do the connected components of acquaintanceship graphs represent? [Ans. Maximal sets of people with the property that for any two of them, we can find a string of acquaintances that takes us from one to the other.

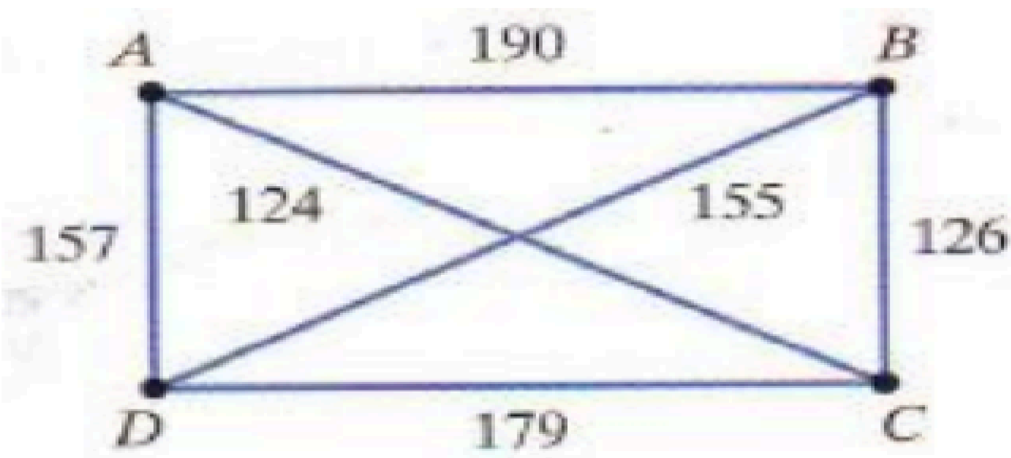
SHORTEST-PATH Problems :

Sales directors for large companies are often required to visit regional offices in a number of different cities. How can these visits be scheduled in the cheapest possible way?

For example, a sales director who lives in city A is required to fly to regional offices in cities B, C, and D. Other than starting and ending the trip in city A, there are no restrictions as to the order in which the other three cities are visited.

The one-way fares between each of the four cities are given in given Table. A graph that models this information is shown in the given Figure. The vertices represent the cities. The airfare between each pair of cities is shown as a number on the respective edge.

	A	B	C	D
A	*	$190	$124	$157
B	$190	*	$126	$155
C	$124	$126	*	$179
D	$157	$155	$179	*



Brute Force Method :

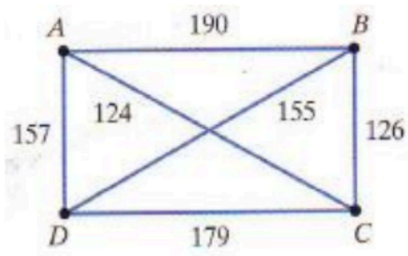
One method for finding an optimal Hamilton circuit is called the Brute Force Method. The optimal solution is found using the following steps:

1. Model the problem with a complete, weighted graph.

2. Make a list of all possible Hamilton circuits.

3. Determine the sum of the weights of the edges for each of these Hamilton circuits. 4.The Hamilton circuit with the minimum sum of weights is the optimal solution

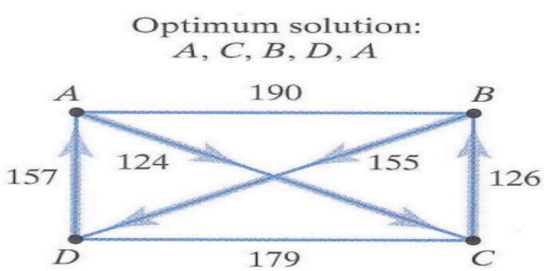
The traveling sales person problem is the problem of finding a Hamilton circuit in a complete weighted graph for which the sum of the weights of the edges is a minimum. Such a Hamilton circuit is called the Optimal Hamilton Circuit or the Optimal Solution.



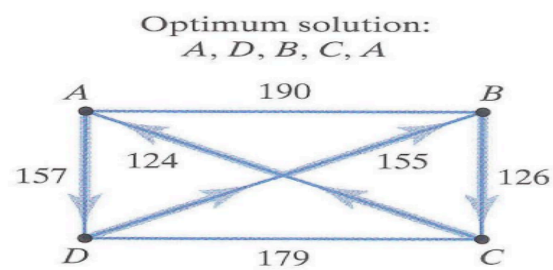
Hamilton Circuit	Sum of the Weights of the Edges	=	Total Cost
A, B, C, D, A	190 + 126 + 179 + 157	=	$652
A, B, D, C, A	190 + 155 + 179 + 124	=	$648
A, C, B, D, A	124 + 126 + 155 + 157	=	$562
A, C, D, B, A	124 + 179 + 155 + 190	=	$648
A, D, B, C, A	157 + 155 + 126 + 124	=	$562
A, D, C, B, A	157 + 179 + 126 + 190	=	$652

Minimum sums

It is clear that there are two Hamilton circuits that have the minimum cost of $562. The optimal solution is either A,C,B,D,A or A,D,B,C,A. For the sales director, this means that either route shown in Figures (a) and (b) has the least expensive way, to visit the regional offices in cities B, C, and D. Notice that any of the two route solution is the reverse of the other.



(a) Fly from A to C to B to D and then back to A



(b) Fly from A to D to B to C and then back to A

Suppose that a supercomputer can find the sum of the weights of one billion, or 109 Hamilton circuits per second. Because there are 31,536,000 seconds in a year, the computer can calculate the sums for approximately 3.2 x 1016 Hamilton circuits in one year. The table below shows that as the number of vertices increases the Brute Force Method is useless even with a powerful computer.

Number of Vertices	Number of Hamilton Circuits	Time Needed by a Supercomputer to Find Sums of All Hamilton Circuits
18	$17! \approx 3.6 \times 10^{14}$	$\approx$0.01 year $\approx$ 3.7 days
19	$18! \approx 6.4 \times 10^{15}$	$\approx$0.2 year $\approx$ 73 days
20	$19! \approx 1.2 \times 10^{17}$	$\approx$3.8 years
21	$20! \approx 2.4 \times 10^{18}$	$\approx$76 years
22	$21! \approx 5.1 \times 10^{19}$	$\approx$1597 years
23	$22! \approx 1.1 \times 10^{21}$	$\approx$35,125 years

The Traveling Salesman Problem was first formulated a in 1930. It is a mathematical problem used in graph theory that requires one to find the most efficient route (tour) that a salesman can take to visit n cities exactly once and return home. In general, the objective is to visit n cities once and return home with the minimum amount of travel. This relates to our project in that we must use the Traveling Salesman Problem in order to find the shortest possible route for a rover that will visit seven sites on Mars. Weighted Graphs And: 4.D Hamiltonian Paths & Circuits Suppose a sales director who lives in city A is required to fly to regional offices in ten other cities and then return home to city A. With(11 - 1)!, or 3,628,800, possible Hamilton circuits, a list is out of the question. What do you think of this option? Start at city A. From there, fly to the city to which the air fare is cheapest. Then from there fly to the next city to which the air fare is cheapest, and so on. From the last of the ten cities, fly home to city A. By continually taking an edge with the smallest weight, we can find approximate solutions to traveling s ales person problems. This method is called the Nearest Neighbor Method. 189 Weighted Graphs and the Traveling Salesperson

Problem 4.D Hamiltonian Paths & Circuits Suppose a sales director who lives in city A is required to fly to regional offices in ten other cities and then return home to city A. With(11 - 1)!, or 3,628,800, possible Hamilton circuits, a list is out of the question. What do you think of this option?

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190 Weighted Graphs and the Traveling Salesperson Problem 4.D Hamiltonian Paths & Circuits The Nearest Neighbor Method Of Finding Approximate Solutions to Travelling Salesperson Problems The optimal solution can be approximated u sing the following steps:

1. Model the problem with a complete, weighted graph.

2. Identify the vertex that serves as the starting point.

3. From the starting point, choose the edge with the smallest weight. Move along this edge to the second vertex. (If there is more than one edge with the smallest weight, choose either one.)

4. From the second vertex, choose the edge with the smallest w eight that does not lead to a vertex already visited. Move along this edge to the third vertex.

5. Continue building the circuit, one vertex at a time, by moving along the edge with the smallest weight until all vertices are visited.

6. From the last vertex, return to the starting point.

Weighted Graphs and the Traveling Salesperson Problem 4.

D Hamiltonian Paths & Circuits A sales director who lives in city A is required to fly to regional offices in cities B, C, D, and E. The weighted graph showing the one-way air fares is given in RHS. Use the Nearest Neighbor Method to find an approximate solution.

What is the total cost?

Solution: The Nearest Neighbor Method is carried out as follows:

1. Start at A.

2. Choose the edge with the smallest weight: 114. Move along this edge to C. (cost:$ 114)

3. From C choose the edge with the smallest weight that does not lead to A:115. Move along this edge to E. (cost: $115)

4. From ,E, choose the edge with the smallest weight that does not lead to a city already visited: 194. Move along this edge to D. (cost: $194).

5. From D, there is little choice but to fly to B, the only city not yet visited. (cost: $ 145)

6. From B, close the circuit and return home to,4. (cost: $180)

An approximate solution is the Hamilton circuit: A,C,E,D,B,A. The total cost is $114 + $115 + $194 + $145 + $180 = $748.